THE UNIVERSITY OF ALBERTA

THE EFFECT OF MODEL BUILDING ON GEOMETRICAL MATURITY

by

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A THESIS

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Abstract

The purpose of this investigation was to examine the effect of studying a unit in solid geometry on the performance of children in a geometrical maturity test.

The experiment was performed in Edmonton, Canada and in Haifa, Israel, with 366 children in grade six, and 75 children in grade seven. In all 21 classes took part in the experiment, the control group consisting of 9 of these classes. A geometrical maturity test was used as the pre-test and post test. In each test four solids were used. The post-test used a different set of solids from the pre-test. This test is based on the work of Piaget and Inhelder (1963) and developed by Boe (1966). After administration of the pre-test, the students in the experimental group in each city studied the unit "Making Rectangular Boxes" (Kuper and Walter, 1978). The post-test was administered to all the students on the completion of the unit.

The main question investigated was: Did the children who studied the unit perform significantly better than those who did not? Two subsidiary questions were examined: (a) Was there any correlation between ability as measured by IQ and the score in the pre-test and post-test and (b) was there any difference between the performance of the children in Haifa and those in Edmonton?

It was found that in Haifa there was a significant

difference between the performance of the experimental group in the post-test and that of the control group. The same effect was observed in Edmonton, but could not be said to be significant. The children in Haifa performed better than those in Edmonton.

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Chapter I

Background to the Problem

1.1 Introduction

Geometry is one of the most neglected subjects in the curriculum of the upper grades of the primary school and the lower grades of the junior high school. And within the frame-work of geometry, solid geometry is the most neglected.

This study will describe a curriculum unit designed to give experience in solid geometry to children in grades six to seven. The material in the unit includes model building and the study of the relation between two- and three-dimensional figures. It gives the students an opportunity to examine bodies in three dimensions and it provides an introduction to the study of volume. In order that the child get the maximum benefit from the exercises in the unit, it is intended that he should discover the relationships by himself or by discussion with his fellow students in the class.

An attempt will be made to see if studying the material described affected the child's performance in a geometrical-maturity test.

1.2 The Traditional Mathematics Curriculum in the Schools

During the first half of the twentieth century, mathematics curricula in most western countries remained static. Some curriculum reforms had been made at the beginning of this century, under the influence of Perry (1902) and Moore (1967).[1] But half a century later, many of the innovations suggested by these critics had still not been implemented. Even in the 1970's, fifty to sixty percent of American schools still use the traditional curriculum (Kline, 1973).

Traditionally, the mathematics taught in primary schools consisted mainly of arithmetic. There was some descriptive geometry, usually to enable students to apply arithmetical calculations to areas, perimeters and volumes. In the junior high school the student studied simple manipulative algebra and arithmetic, whilst the senior high school course consisted of algebra, deductive geometry and simple trigonometry. The geometry course, i.e. Euclidean geometry, was taught in the tenth grade only, in schools in the United States. Solid geometry, if taught at all, was part of the twelfth-grade curriculum.

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[1]Moore's address before the American Mathematical Society in 1902 was reprinted in the Mathematics Teacher as one of a series of articles entitled 'Classics in Mathematics Education' (Moore, 1967).

In Britain, the traditional curriculum in the grammar school comprised arithmetic, geometry, algebra, and trigonometry. The first three subjects were taught in parallel throughout the first five years of the grammar school. In the last two years, in the "sixth form", the student studied calculus, analytic geometry, algebra and in some cases applied mathematics. However, only students intending to study sciences in the university would continue to study mathematics in the sixth form.

The curriculum in mathematics in Israel schools was in most respects similar to that in Britain. However the age grouping in the schools was structured differently. Until the 1968 reform in the educational system, eight years of primary school were followed by four years of selective high school. The high-school curriculum led to a final level of knowledge similar to that in Britain. After the reform in the school system, the age grouping was converted to six years in the primary school, followed by three years in junior high school and three years in the senior high school or similar school.

To illustrate the methods used in teaching mathematics in the traditional curriculum, it is instructive to examine the text-books used in the schools. One of the most widely used series of text-books in Britain (and its Dominions and Colonies) was that of Durell. Durell's General Arithmetic for Schools (Durell, 1959) was first published in 1936 and by 1959 it had been reprinted 29 times. The opening sentence in the introduction is reveal-

ing:

> The character of this book has been determined by the belief that the *primary object* in teaching elementary arithmetic is to ensure accuracy. In pure computation the less the pupil has to think, the more likely that mistakes will be avoided.
>
> (Durell, 1959,p.vii, my italics)

In 1978 the attitude represented by this last statement cannot be considered as desirable or valid.

1.3 The Teaching of Geometry

The geometry taught in the high school followed essentially the sequence of Euclid's Elements. This was true irrespective of whether the course was taught intensively for one year (as in North America), or for several years, in parallel with algebra and arithmetic (as in Britain). The Israeli system was similar to that of Britain.

Euclidean geometry has formed part of the education of the young since the time of the Greeks. The classical education, the quadrivium, consisted of the study of logic, philosophy, harmony and geometry. The rationale for teaching geometry has always been its emphasis on deductive reasoning and mathematical rigour. However this rigour is quite often pseudo-rigour:

> 'Due to the complicated nature of Euclid's approach and to a certain obscurity in the foundations, a capricious mixture of rigour and intuition penetrated the teaching of geometry. Some textbooks listed as many as forty axioms for elementary geometry, while leaving the axiomatic

> structure of the basic geometrical concepts wide open so that again and again intuitive arguments had to be appealed to.'
>
> (Steiner, 1970)

The drawbacks and logical defects of Euclidean geometry are discussed by Klein (1932). One of Klein's criticisms of the "Elements" of Euclid is the rigid separation of plane and solid geometry. In addition Klein discusses the lack of rigour in some of Euclid's definitions. For example, the concept of a rigid translation is tacitly invoked in the treatment of congruence. Another shortcoming in Euclidean geometry is the almost complete disregard for order concepts (e.g. "between", existence of two sides of a line, the interior of an angle) which are not defined in the postulates. Moreover, some of the constructions evoke tacit assumptions: for example, Euclid assumes that some lines and circles intersect in points.

The fifth axiom of Euclid has been a source of inquiry for mathematicians throughout the centuries. This axiom states: "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles." (Euclid, trans. 1956). The axiom is nowadays usually quoted in Playfair's version: "Through any given point not on a given straight line there can be drawn only one line parallel to a given line."

The axiom as stated by Euclid is much more complex in form than the four previous axioms stated in the Elements. Mathematicians felt that there was a need to search for a proof of the fifth axiom, which would be dependent on the four earlier axioms, thus dispensing with the need for the fifth axiom. The search for this proof forms a fascinating part of the history of mathematics.

Meder (1958) points out that a treatment of geometry which evolved classically to train intellectuals and philosophers is not necessarily the best way to produce mathematicians. Nor is it necessarily the best way to teach mathematics to children. He claims that understanding is lost when mathematics is embedded in logical formalism and when emphasis is placed on the axiomatic structure. He states that instead:

> Our aim should be to encourage our students to approach the study of geometry in a spirit of discovery and adventure, to lead them to understand to the fullest extent consistent with their ability and maturity how mathematical facts are discovered and how mathematical concepts are formulated and may be generalized and extended.
>
> (Meder, 1958, p.583)

From the arguments of these writers, we see that (a) Euclid's logic is incomplete and that (b) even if it were not, the emphasis on logic is at the expense of understanding the content. These criticisms of Euclidean geometry have serious implications for the teaching of geometry in the secondary-school classroom.

1.3.1 The Premature Pursuit of Rigour at the Expense of Physical Understanding

The emphasis on rigour is a defect not only of the traditional geometry curriculum but of many of the new curricula as well. Kline (1958), in an article critical of the teaching of modern mathematics in the high school, attacks the tendency to prematurely demand rigour in mathematical proofs, definitions and structures, before the student has the capacity to understand and appreciate the meaning of rigour. He says:

> Mathematics must be understood intuitively in physical or geometrical terms. This is the primary pedagogical objective. When this is achieved, it is proper to formulate the concepts and reasoning in as rigorous a form as young people can take.
>
> (Kline, 1958, p.423)

Definitions and postulates with this emphasis on rigour can be found in many textbooks. For example, in the junior-high-school textbook by Van Engen et al. (1964), the concept of area is introduced as follows:

> The union of a simple closed curve and its interior is called a closed region...A measure of a closed region is a unit closed region... The number of unit closed regions that do not overlap and that can be included in a closed region is the area of the closed region.
>
> (Van Engen et al., 1964, p.469)

The textbook is one written after the introduction of the "New Math". The definition quoted above is a mathematically correct and rigorous definition of area. However area has a physical meaning — it is a measure of the

surface. Van Engen's formal definition of area does not give the child any understanding of the physical meaning of area.

1.3.2 Intuition in the Teaching of Geometry

The new approach which is needed in the primary and junior high schools should stimulate "intuitive" understanding of geometry. By an intuitive understanding is meant that the child can give a correct answer to a question without necessarily being able to give an analytic explanation for his reply. Indeed he may give a correct response, without being able to give any explanation at all. For example, he may "know" what is meant by a square. He can correctly pick out a square from a collection of models or drawings of a set of quadrilaterals, but when required to describe a square, the child may not be able to formulate a set of sufficient (or even necessary) criteria.

In addition to this intuitive knowledge, students need opportunities to visualize and to have physical contact with geometrical bodies and shapes. This is desirable both on psychological as well as on pedagogical grounds, as will be shown in the second chapter of this report.

One of the recommendations of the Cambridge Conference on School Mathematics reported in *Goals for School Mathematics (1963)* stresses the need for a more intuitive approach. Similar pleas were made in the report of the

Royamont Seminar, held under the auspices of the Organization for Economic Cooperation (1961), the K - 13 Geometry Committee of the Ontario Institute for Studies in Education (1967), and in the report of the British Schools Council for the Curriculum and Examinations curriculum bulletin entitled *Mathematics in Primary Schools* (H.M.S.O., 1965).

1.3.3 Three-dimensional Geometry

There is an almost complete neglect of the three dimensionality of the world in the teaching of geometry in the schools. The child lives in a world of three dimensions and not of two. Yet most of the geometry he learns at school is in the simpler and more abstract world of two dimensions. Furthermore, most of the exercises and examples are performed as paper and pencil activities. Throughout most of their school experience in the geometry classroom, the students never have to look at lines, planes and bodies in a three-dimensional setting.

In Alberta, the textbook in common use for the geometry high school course is Wilcox (1968). Out of its 376 pages only 34 are devoted to solid geometry (i.e. about 9%). Even where solid geometry is studied in the last year of the senior high school by the few mathematics specialists the exercises set are very contrived. A typical example is:

> In triangle *ABC* , *AB* = *BC*. *BN* is a line segment perpendicular to the plane *ABC*. A plane through the mid-point of *BN* intersects *AN* at *P* and *BN* at *Q*. Prove that *ABQC* is an isosceles trapezoid.
>
> (Weeks and Adkins, 1961, p.172)

Without a suitable diagram, an ability to visualize in three-dimensional space, and a considerable skill in perspective drawing, the student cannot even begin to apply his skill in logical deduction to this type of problem.

However, even fewer students are studying solid geometry than in the past. There is a tendency nowadays to choose other options such as probability and statistics or computer sciences.

1.3.4 The Emphasis on Right Solids

A few simple, regular right[1] solids form almost the entire content of the senior-high-school solid-geometry course. Non-right and irregular solids are neglected. These solids (see Fig. 1) include the prism whose base is not an equilateral triangle, square or rectangle. It also includes the solids which are not right solids, such as the oblique parallelepipeds[2].

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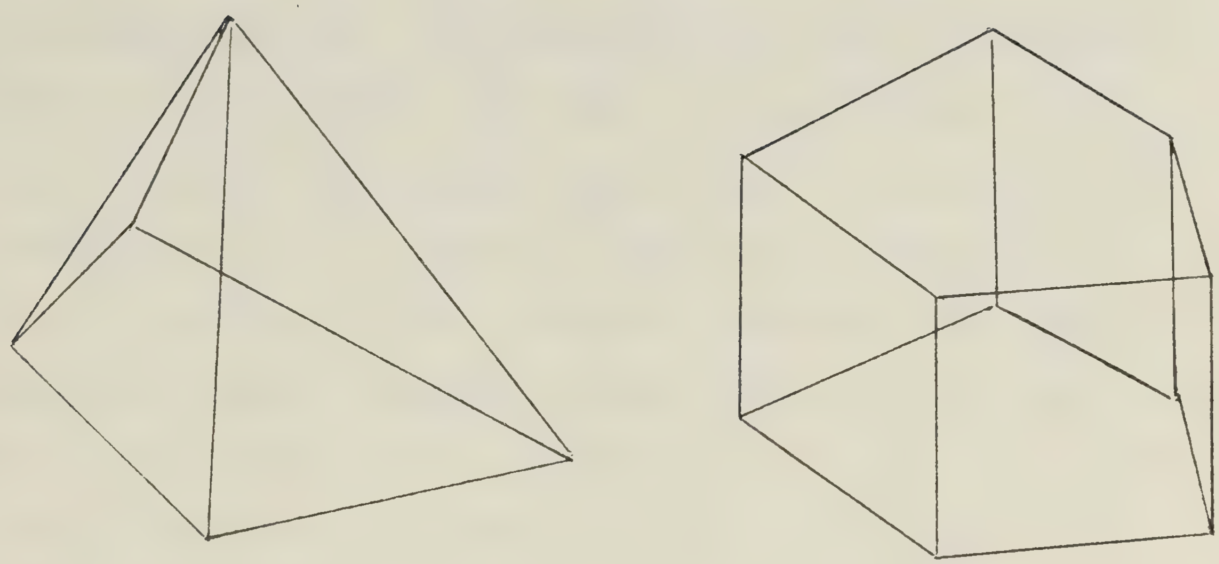
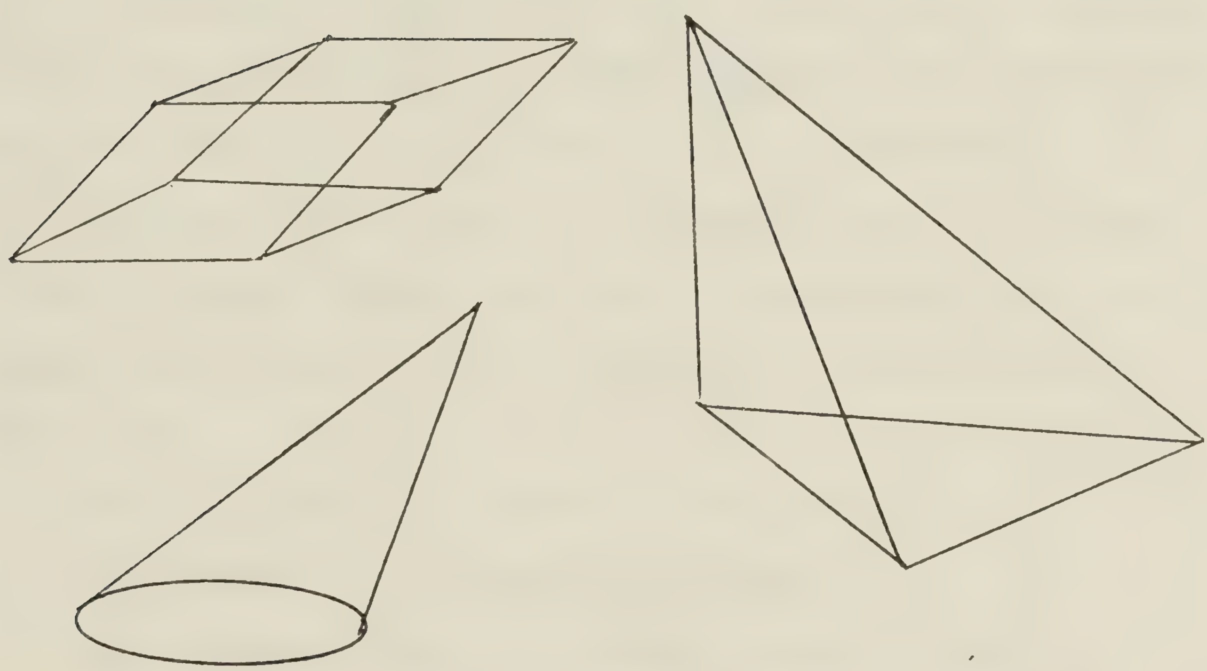
[1]A right solid is one in which the ends or bases are at right angles to the axis of the solid.

[2]A parallelepiped is a six-faced solid, with three sets of parallel faces, each pair of which are congruent parallelograms.



Right irregular solids



Skew solids

Figure 1

Right Irregular Solids & Skew Solids

Even in those schools which include solid geometry, the students rarely meet the five regular Platonic solids, and, even more rarely, the thirteen semi-regular Archimedean solids. The students learn that only five Platonic solids exist, and that these were known to the Greeks as symbols of perfection. They learn that Euler's formula $F + V = E - 2$ holds. (Here F is the number of faces of the body, V the number of vertices and E the number of edges.) However students hardly ever have a chance to explore other types of solids and to show that similar relationships hold for these solids too.

1.3.5 Measurement in Geometry

The study of geometry in the schools can be separated into two main catagories, namely, the study of the properties of the shapes and solids and the measurement of parts of the shapes and solids. The former involves the knowledge of names, properties and relationships of the various geometrical figures. For example, the student should know that:

(a) a square is a special type of rectangle,

(b) not all rectangles are squares,

(c) the square is a form of rhombus,

(d) the diagonals of a square are equal,

(e) the diagonals of a square bisect each other at right angles, and

(f) the square has four axes of symmetry.

Most school curricula also include the study of measurement of shapes and solids. This includes calculation of perimeter, area and volume. It is clear that in order to perform anything more than the most elementary calculations, a knowledge of the properties, or at least some selected properties, of the shapes is required.

One of the challenges in teaching elementary mathematics is the difficulties children experience in distinguishing between perimeter, area and volume. The problem is well known to school teachers. Lovell (1971) in a paper dealing with some spatial ideas of school children points out that this ".. is an issue well-recognized by elementary school teachers, that children confuse perimeter and area."

One likely reason for the confusion is that the concepts of perimeter, area and volume are introduced in quick succession (usually in the upper grades of the primary school). The child does not always have time to assimilate one of these ideas before he meets the next one. As the subject is usually taught, three distinct skills are required:

(a) an understanding of the concepts of perimeter, area and volume, and the distinctions between them,

(b) an ability to perform the necessary arithmetical calculations, and

(c) an understanding of the different units of

length, area and volume.

Perimeter is measured in centimetres, inches, metres. Area is measured in square centimetres (cm^2), square feet or acres. Volume is measured in cubic inches, cubic centimetre (cm^3) or cubic metre (m^3). There are not sufficient differences between these names for the child to easily distinguish between them. It is even more difficult for him to convert from one set of units to another.

Another potential source of confusion is that, for example, a rectangle may be drawn to represent the perimeter of a field, but in another problem the same rectangle may represent the area of a table. Thus in

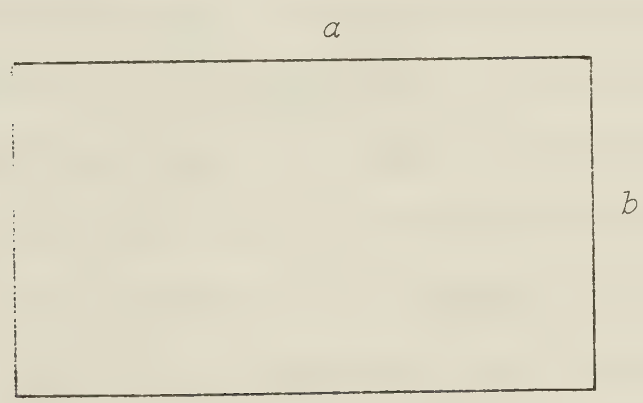


Figure 2

Perimeter or Area?

Fig. 2, given the lengths of the sides of the rectangle as a units and b units, the child has to calculate $a + b + a + b$, or perhaps $2a + 2b$, or even $2(a + b)$ units in order to find the perimeter. To find the area he has to calculate $a \times b$ square units. Although he has the necessary skills in both addition and multiplication, unless he is sure of *what* he is doing, he will not know *when* to multiply and when to *add*.

When a child begins to learn volume his confusion is compounded. The algorithm usually taught for the volume of a rectangular prism is to multiply the length by the breadth by the height. But which measurement is the height and which is the breadth? Are not all these terms equivalent to length?

It is therefore important that the child be presented with appropriate physical experiences of perimeter, area and volume before being confronted with exercises in calculation. Here again pen and paper exercises are not sufficient. If the child sees a two-dimensional drawing of a three-dimensional figure on the blackboard, it cannot be assumed that the child recognizes the drawing as a perspective representation of the solid body. He has to understand that the figure is a solid before he can perform the necessary calculations.

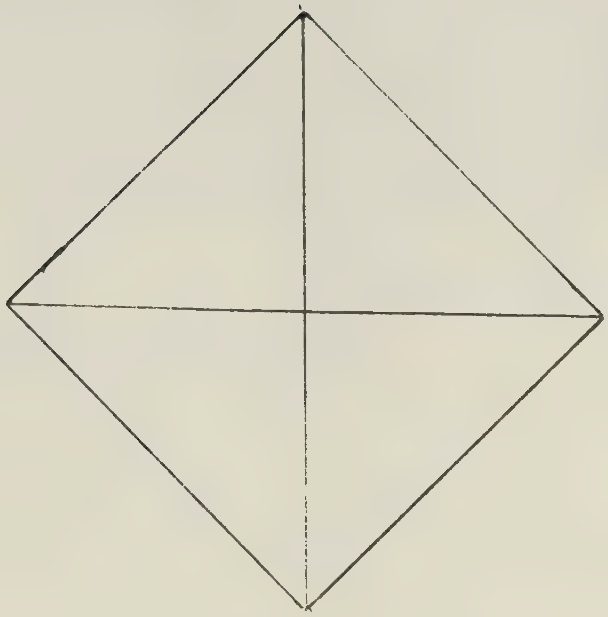


Figure 3

Bishop's Diagram

In a recent article, Bishop (1977) describes an experiment in which he showed some Papuan students Fig. 3. He asked them to make the object shown in the diagram from cocktail sticks and plasticine. Some of the students produced a square with two diagonals, and others a square-based pyramid. Both these "wrong" interpretations are as legitimate interpretations of the plane diagram as the intended one of a regular tetrahedron. Difficulties were also experienced in what Bishop calls "the standard and conventionalized representation which makes little concession to reality". This is particularly true when we consider the drawings presented to children in the geometry class. He continues:

> If you now consider the number of diagrams, graphs, sketches, constructions and tables which we use in mathematics, is it not strange how little time we give to techniques which must be mastered, to learn the complex visual vocabulary which must be learnt and to the many arbitrary conventions and rules which must be digested?
>
> (Bishop, 1977)

1.4 "Making Rectangular Solids"

"Making Rectangular Solids", the unit discussed in this report, was prepared as an attempt to provide children with experience in solid geometry. It is intended to be an introduction to the study of volume, especially of the rectangular prisms. It provides opportunities for the child to build, examine, handle,and draw the solids, which he will deal with in his first contact with the study of volume. He will be able to compare drawings of the solids with the actual bodies, to transform a two-dimensional net of a solid into the solid itself, and to examine the relationship of some particular lines in space with the solids bodies he has already built.

It is hoped that the unit will help the student in the upper primary or lower secondary grades to develop maturity in looking at geometrical solids and in relating those solids to the appropriate drawings and plans of the solids.

Chapter II

Review of the Literature

2.1 The Psychological Understanding of Space and Geometry

2.1.1 The Piagetian Theory

Most present-day studies on the psychological understanding of space and geometry are influenced by two books: *The Child's Conception of Space* (Piaget and Inhelder, 1963) and *The Child's Conception of Geometry* (Piaget, Inhelder, and Szeminska, 1960). According to Piaget and his coworkers, the child's understanding of space unfolds in a hierarchial pattern.

The first impressions the baby forms of the world are "haptic" or tactile in nature. The relationships the child perceives are purely qualitative ones: proximity, separation, order (spatial succession) and enclosure. These relationships are non-metric in character; they are basic to the branch of mathematics called topology. At this stage the baby has no knowledge of distance or angle, nor of the permanence of objects.

Later the child acquires the concept of projective space. An object is no longer seen in isolation but is perceived in relation to other objects. At about the age of seven, the child can build a straight line, irrespective of the frame of reference. The ability to coordinate perspectives and to develop a less egocentric thought

structure appears later (Flavell, 1963).

In the third stage (from seven years onwards) the child acquires a knowledge of Euclidean space (Piaget and Inhelder, 1963). He begins to develop a two-dimensional framework. Moreover, he is now able to conserve substance and length. The concept of conservation is essential for the child to learn to measure. In order to use a measuring instrument the child needs the ability to conserve and in addition needs to understand the concept of reversibility, i.e., he understands that if $A = B$, then $B = A$. In addition he has acquired the concept of transitivity, namely, if $A = B$ and $B = C$, then $A = C$.

From about the age of 12 onwards, most children understand the properties of number, space, and time. The child can usually construct formal logical theories, and can reason, deduce and hypothesize. He has acquired the concept of continuity, which is a prerequisite for the full understanding of volume.

Piaget sees the understanding of volume as developing in two main stages. At the "concrete-operational" stage, between the ages of 7 and 12, the child's ideas of volume are topological as well as Euclidean. His ability to conserve volume refers only to the interior volume, i.e., to the volume inside a bounding surface (Copeland, 1974, p. 301). The interior volume is *not* the same as the volume actually occupied by the body, namely, the volume of the liquid it would displace if submerged.

It is only when the child has reached the formal operational stage that he understands that this "total" volume is conserved:

> Having understood the relation between boundary lines (or areas) and interior volume, children inevitably deepen their sense of the conservation of volume. They discover for the first time that it is not merely the interior "contained" which is invariant but the space occupied in a wider context.
>
> (Piaget et al., 1960, p. 385)

It is interesting to note that the development of the understanding of space by the child, starting from topological space, through projective to Euclidean space, is in the reverse order to the historical development of geometry.

2.1.2 Experimental Studies

Lovell and his colleagues, in a series of studies in England in the 1950's and 1960's, repeated Piaget's experiments in a more rigorous and controlled manner. Lovell (1959) found partial agreement and partial disagreement with Piaget's results. In particular, the variation in performance within a particular age group was greater than Piaget suggests.

Lovell's group reported studies on the conservation of substance (Lovell & Ogilvie, 1960), the conservation of volume (Lovell and Ogilvie, 1961), and the growth of geometrical concepts (Lovell, Healey and Rowland, 1962).

Lovell and Ogilvie (1961) found that the concepts of interior volume, of occupied volume and of displacement

volume are acquired slowly by children. They are not completely subsumed until the child is about 12 years old.

Elkind (1961) systematically repeated Piaget's investigations into the ages at which children conserve quantity, weight and volume. He tested 175 children aged from five to eleven. There were 25 children in each of the 7 age groups. Of the children in the eleven-year-old group, only 25% (6 - 7 children) were conservers. Elkind's conclusions are in close agreement with Piaget's findings that there is a regular age-related order in reaching conservation of volume. However he disagrees with Piaget about the actual age at which each stage is reached. The size of Elkind's sample - in the opinion of the present author - does not seem adequate to warrant drawing such sweeping conclusions.

Beard (1962) studied the order in which concepts are developed. She performed a series of experiments using different types of materials. She finds that for the children in her sample, Piaget's order of conservation (quantity, weight and then volume) was not borne out. She suggests that experience rather than maturity is the factor which determines the order of attainment of the concepts. Volume conservation was reached at a later stage than was stated by Piaget. Beard states that this was probably due to the fact that the children in her sample had not been exposed to suitable experiences.

However it is difficult to accept all the conclusions of Beard. In the four experiments described in Beard (1961),

the number of students in the age group of interest to this study (10 to 11 years) was small. Thus for the four experiments listed in Table 1, the number of conservers was 10, 15, 3 and 12 respectively.

The author is particularly sceptical about Beard's third experiment, using salt in water (Table 1, Beard (1962) c). This experiment is of a very different nature from the others. The child is asked questions about the change in level of water in a glass when salt is dissolved in it. The solid used here is soluble in contrast to the plasticine or the torch (flash-light) battery used in Beard's other experiments. These other tests use the physical properties of liquids to estimate volume. But the solution of salt in water involves chemistry as well as physics. Many adults might have difficulty predicting the answer to this problem.

Conservation experiments were performed by Vinh-Bang and Inhelder (1962). Again the size of the sample involved was small. 27 students between the ages of 8 and 12 were examined. 80% of the eleven-year old children in this sample were found to be conservers.

Uzgiris (1964) used four different deformable materials, as shown in Table 2. These were shown to the child in succession. He was asked to predict the level of the water in a vessel when each was immersed in it. Only if the child responded correctly to *all* the situations was he considered to conserve volume. Of the twenty grade-6 children in Uzgiris's sample only 5 were conservers of volume. He con-

Table 1

Experimental Studies on Conservation of Volume

Experimenter	Material Used	Percentage of correct responses: 9-10 yrs	11 yrs	12 yrs
Lovell and Ogilvie (1961)	Wooden blocks and water	60	80	-
Elkind (1961)	Plasticine ball and "sausage", in water	19	25	-
Beard (1962)	(a) Plasticine ball and "biscuit", in water	31.2	-	-
	(b) Plasticine and ping-pong ball, in water	9.4	-	-
	(c) Salt in water	48.4	-	-
	(d) Torch battery in water	37.5	-	-
Vinh-Bang and Inhelder (1962)	Cylinders of different weights in water	15	37	48
Uzgiris (1964)	Plastic balls, towers of nuts, wire coils, plastic wire	15	15	20
Vernon (1965)	Plasticine and water	45	-	-

Table 2

Forms of Presentation of the Materials
in the Experiment of Uzgiris (1964)

MATERIAL	PLASTICINE	18 METAL NUTS	COIL OF WIRE	PLASTIC CORD
SHAPE				
1.	ball	tower 3x2x3	coil	straight length
2.	sausage	tower 3x3x2	slightly stretched	simple knot
3.	long cylinder	tower 3x1x6	stretched almost straight	second knot twisted around the first
4.	three pieces	three piles 3x2	two pieces each containing one third of the strands	cut into three pieces

cludes that conservation of quantity, weight and volume is reached in that order. There are situational differences and inconsistencies across the materials which may be due to individual past experience. However, conclusions are arrived at by generalizations based on a small sample of children.

Vernon (1965) examined 100 boys aged between 10 and 11 years. He used two equal balls of plasticine and two jars of water. One of the balls of plasticine was made into a disc. Only 45% of the children were able to answer correctly when asked to predict what would happen when the disc of plasticine was put into water.

Table 1 compares the results of the conservation experiments described above. It is clear that not all children have understood conservation of volume by the age of 11 or 12 years. This seems to be a transitional age. Slight differences in the way of presentation of the test seem to influence the results greatly. Nor can the previous educational background and experience of the child be discounted. There is evidence that suitable concrete experiences can aid the child to reach the state of conservation of volume earlier than he would have done if the experience had not been provided.

The patterns of development described by Piaget are being developed and reviewed in the light of experience and understanding. In 1963 Dodwell claimed that the patterns of development of conservation are more complex and less clear-

ly demarcated than Piaget suggests. In a later paper (Dodwell, 1971), he claims that little advance has been made in the psychology of a child's perception and understanding of mathematical ideas. He points out that concentration of the study of visual pattern recognition, of perceptual coding and the biological nature of perceptual learning, might give more insight into our knowledge of the child's understanding of space.

The unit which is tested in this report was designed to give the child concrete experience as a prelude to the study of volume. Instead of just studying the algorithms for calculating volumes, the unit aims to provide experiences in examining solid bodies, before performing the usual calculations. In particular the child builds and plays with a set of rectangular prisms, looks closely at them and classifies them in various ways. He examines the relation of some non-planar lines to the solid, as well as looking at the different ways a rectangle can be put in a cube. Estimation of volume relationships precedes calculation of volume. This is first attempted by filling the prisms with unit cubes.

It is hoped that these concrete experiences will accelerate the understanding of the conservation of volume, and the attainment of geometrical maturity, as defined in Chapter IV.

2.2 The Geometry Curriculum in the Schools

The motivation for writing the unit "Making Rectangular Solids," (on which this report is based) was, in part, the need to improve the geometry curriculum in the schools.

Geometry has been one of the most neglected sections of the mathematics curriculum. This review shows that the need for improvement has been apparent for a long time. Although many attempts have been made to improve the situation, not a great deal has in fact been changed. Changes made in the curriculum, and in attitudes of educators and their committees, have in fact not penetrated to most of the teachers in the classroom (see sections 2.3 and 2.4).

2.2.1 Review of the Changes in the Geometry Curriculum

Since the introduction of universal education in the western world, the classical geometry curriculum has been under attack. Griffith and Howson (1974) review the history of the growth of the various professional associations (The National Council of Teachers of Mathematics in North America and the Mathematical Association in Britain) which were founded primarily to attempt this curriculum reform. Griffith and Howson discuss the success of these reforms.

Many of the early proposals for reform recommended that before formal axiomatic geometry was taught, children should be exposed to informal geometrical concepts. These proposals were based on the work of such educators as

Froebel, Pestalozzi and Herbart. The early-education programmes of these pioneers included the use of toys, geometrical bodies, as well as other manipulative materials (National Council of Teachers of Mathematics, 1973). Both Perry (1902) and Moore (1967) advocated the more practical approach to the teaching of geometry. Perry states that it is

> essential that the student be thoroughly familiar through experiment, illustration, measurement and every other possible method, with the ideas to which he applies his logic.
>
> (Perry, 1902)

In 1912, the U.S. National Committee of Fifteen on Geometry published their final report (Mathematics Teacher, 1912). They recommended that a special course of geometry was desirable in the seventh and eighth grades. In addition they stressed that

> Contrary to traditional procedure, the forms of solid geometry should be emphasized even more than those of plane geometry, for they are more real and capable of concrete illustration.
>
> (Report of the National Committee of Fifteen on Geometry, 1912)

In the 1960's and after, many important changes have been made in the mathematics curricula. In Britain the study of mathematics in the primary school was greatly influenced by the report *Mathematics in the Primary Schools* (1965), and by the work of Biggs and Maclean (1969), Dienes (1964) and by Nuffield Mathematics Project (1967).

The emphasis was placed on a curriculum in which geometry is integrated with arithmetic and algebra. These authors contend that geometry should be taught through "motion geometry". This approach, through the symmetries and isometric transformations was first advocated by Klein in his Erlanger programme of 1872 (Klein 1932). The newer mathematics programmes in the high schools in Britain (School Mathematics Project, 1965, Scottish Mathematics Group, 1971, Mansfield and Thompson, 1962) adopt this point of view.

These new projects also emphasized that geometry should be taught through active participation of the student:

> 1. Children learn mathematical concepts more slowly than we realized. They learn by their own activities.
> 2. Although children think and reason in different ways they all pass through certain stages depending on their chronological and mental ages and their experience.
> 3. We can accelerate their learning by providing suitable experiences, particularly if we introduce appropriate language simultaneously.
> 4. Practice is necessary to fix a concept once it has been understood, therefore practice should follow, and not precede, discovery.
>
> (Report of HMSO, 1965)

In the U.S. the pioneering work in the field of mathematics education was influenced by Beberman and the Schools Mathematics Study Group (1961), the University of Illinois Committee on School Mathematics (1961), and by the work of the Cambridge Conference on School Mathematics (1963). In Canada an important report was that of the K-13 Geometry Committee of the Development Division of the Ontario

Curriculum Institute (The Ontario Institute for Studies in Education, 1967).

In all these reports and new curricula, stress is laid on the need to teach geometry in a different way from the formal deductive-axiomatic framework of Euclidean geometry. Geometry should develop the spatial intuition of the child, should include concrete activities such as constructing three-dimensional solids, paper-folding exercises, the use of geoboards, geostrips and curve stitching. The new material should include the isometric and other transformations to study the properties of shapes and especially their symmetries.

One of the innovations in many of the reforms has been the recognition that the psychology of learning is at least as relevant as the mathematical content (Skemp, 1971; Biggs, 1967; Copeland, 1976). Copeland (1974) proposed that the teaching of mathematics could be improved if we regard some of the ideas of Piaget as the basis of diagnostic tools. Indeed the Nuffield Project in England uses some of these ideas as guides to teachers for checking the progress of the student (Nuffield Foundation, 1970). In teaching geometry, Copeland maintains that the introduction of formal definitions of such concepts as sets of points, rays, segments and angles are useless rote learning in the early grades of the primary school. He advocates designing the mathematics curriculum to suit the Piagetian level of maturity of the child. There is, therefore, a need for individual-

ization of instruction to allow for the wide range of mathematical maturity to be found in any one-age classroom.

The unit "Making Rectangular Solids" (Kuper and Walter, 1978, Appendix 1) is one of a series of units written to improve the geometry curriculum. It is hoped that the material is presented in a way which will overcome the criticisms listed in the previous paragraphs. The presentation is such that the child is obliged to build, to play and to handle concrete materials. It enables him to operate at his own level of maturity, yet there are opportunities for the more able students to explore mathematical problems.

2.2.2 Geometry Education in Israel

The Israel school system is sub-divided into the Jewish, the Arabic (Moslem) and the Independent (usually Christian) systems. The Jewish schools are further subdivided into the secular and the religious streams. It is to the largest stream, the Jewish secular system, that the following notes will refer. The Israel mathematics curriculum resembles the British curriculum more than it does the North American one.

Under the British Mandate, the mathematics curriculum had been similar to that prevailing in Britain. Many immigrants from Eastern Europe arrived in Israel in the 1930's and 1940's. Amongst them were good and experienced teachers of mathematics. Many of these teachers, of German,

Rumanian and Hungarian origin, brought with them the strong classical traditions of those countries. The geometry curriculum was classical Euclidean geometry. It was taught in the ninth and tenth grades, as well as forming part of the school-leaving examination, which was taken in the twelfth grade. This curriculum included solid geometry.

Since 1969 a reform in the school system has been introduced. Instead of an eight-year primary school, followed by a four-year high-school programme, a six-year primary school programme is followed by three years in the junior high school and three years in the senior high school. These last two schools usually form a single administrative unit. However in the course of the eight years since the first changes were made, only about 50-60% of the schools have in fact been included in the reform. Considerations of local political expediency have slowed the adoption of the reform.

The official curriculum of the Ministry of Education for the reformed part of the school system includes geometry as half of the ninth-grade curriculum. The aim of the curriculum is to teach just sufficient geometry to enable the non-specialist mathematics student to understand simple trigonometry of the right-angled triangle. More advanced students can choose a mathematics option in the eleventh grade, with an additional option in solid geometry (Rehovot, 1970).

In 1972 a new primary-school curriculum in mathematics

was introduced. Previously the amount of geometry in the curriculum was insignificant. The geometry in the new curriculum constitutes 25% of the total mathematics in each grade. The recommended approach is through transformation geometry (reflections, rotations and translations) and by the study of the symmetries of geometrical figures, and not by the study of Euclidean geometry.

In order to effect such a radical reform, the Curriculum Centre of the Ministry of Education had to produce suitable material for the schools (Walter, 1974; Mountwitten, 1977; Kuper and Walter, 1976, 1978). The rationale and methods of this reform and the work of the Centre are described by Eden (1974, 1978). The unit described in this report (Section 4.3.2) forms part of this material. It is part of the sixth-grade geometry curriculum in solid geometry.

At the same time it was decided to reduce the formal "blackboard and chalk" type of instruction, and to try to introduce more activity-oriented methods of instruction. The extent to which this has succeeded will be discussed in the next section (Tehori, 1977).

2.3 Non-implementation of Curriculum Reform Proposals

This section reviews three representative studies on the teaching of geometry after the introduction of curriculum reform. The studies were made in different places and at different levels of the school system. No attempt has

been made to be comprehensive. It is clear that, in each case, recommended reforms have not been effective, nor have the reforms been popular among the teaching fraternity.

2.3.1 The Mathematics Achievement Test for the USA College Board

Williams (1970) reported on an American survey made in the year 1965-66. She examined the implementation of the recommendations of the 1959 "Commission for College Preparatory Mathematics", set up by the USA College Board and the extent to which the recommended changes had begun to appear in the school curriculum. The questionaire was completed by 2718 seniors who had taken the mathematics achievement test for the USA College Board admissions programme.

The Commission's nine-point programme was intended for students capable of doing college work. Only one of the nine points referred to geometry. It was recommended that the geometry programme should contain plane geometry, with coordinate geometry, and in addition, the essentials of solid geometry and space perception. About one third of a semester should be devoted to solid geometry.

As can be seen from Table 3, only a few students had studied geometry in accordance with the Commission's recommendations. Only 19% of the courses dealt with deductive solid geometry, while 13% dealt with intuitive solid geometry. Neither the *proportion* of the course, nor the *time* spent on these subjects was reported, but only the bare fact

Table 3

Percentage Distribution of Description of the First Full-year Course in Geometry

Description of content	Percent of total sample
The course dealt almost entirely with *deductive plane geometry*. Solid geometry in any form and coordinate geometry received little or no attention.	35
The course dealt mainly with both *deductive plane* and *coordinate plane geometry*. Solid geometry received little or no attention.	28
The course dealt mainly with *deductive plane geometry* and *deductive solid* geometry. Coordinate geometry received little or no attention.	8
The course dealt mainly with *deductive plane geometry* and *intuitive solid geometry*. Coordinate geometry received little or no attention.	5
The course dealt in some detail with three types of geometry -- *deductive plane, deductive solid,* and *coordinate*.	11
The course dealt in some detail with three types of geometry -- *deductive plane, intuitive solid,* and *coordinate*.	8
Did not take a full-year course in geometry.	2

Note: Rounding errors and no response account for percentages not adding up to 100%.

that they were included, or not included, in the course. However it is clear that 68% of this sample of future university mathematics students had not studied *any* solid geometry (Williams, 1970).

2.3.2 Geometry in the High Schools in Edmonton

During the winter of 1974-75, the members of a research seminar in the University of Alberta, Edmonton, sent a questionnaire to teachers of mathematics in the high schools in the Edmonton area. The seminar was under the guidance of Sigurson of the Department of Mathematics Education. The purpose of the survey was to examine the state of the teaching of geometry in the area. The teachers were asked about the amount of time actually spent on geometry in the classroom and on the amount of time that they would prefer to spend on geometry. They were also asked to comment on the curriculum then in use. The results of that survey are summarized in Table 4 (private communication).

It was clear, both from the time spent on geometry, and from the teachers comments, that they did not use - or even want - the new curriculum. Not one of the schools in the sample spent as much time on geometry as was recommended by the Department of Education in Edmonton. The teachers' attitudes to the teaching of geometry in the tenth-grade course, (Math 10), were examined. An overwhelming majority of the teachers considered that parts of the course were too difficult; nevertheless most preferred to spend even

Table 4

The Teaching of Geometry in the Edmonton Senior High School

A Survey of Teachers, 1975

	Fraction of time spent on geometry in the mathematics course	Fraction of the schools spending that time
1. MATH 10	15%	10% - 25%
	12%	More than 25%
	1%	Less than 10%
Recommendation of the Department of Education, Edmonton	40%	
2. MATH 20	19%	10% - 25%
	4%	Less than 10%
	3%	More than 25%
Recommendation of the Department of Education, Edmonton	33%	

less time on geometry. Indeed, they felt that only 10% of the mathematics time should be devoted to geometry.

The report of this survey indicated that students in Edmonton high schools leave school having studied algebra, but little or no geometry, in their mathematics courses.

2.3.3 Geometry in the Primary School in Israel

A report on the state of mathematics teaching in the primary school in Israel has recently been prepared (Tehori 1978). The survey on which it was based was conducted in 1976. The information on the geometry curriculum is quoted in Figures 4 to 7. Information is not yet available for grades five and six, but the general picture is not obviously different from grades one to four. The details for some of the arithmetical concepts are provided for comparison.

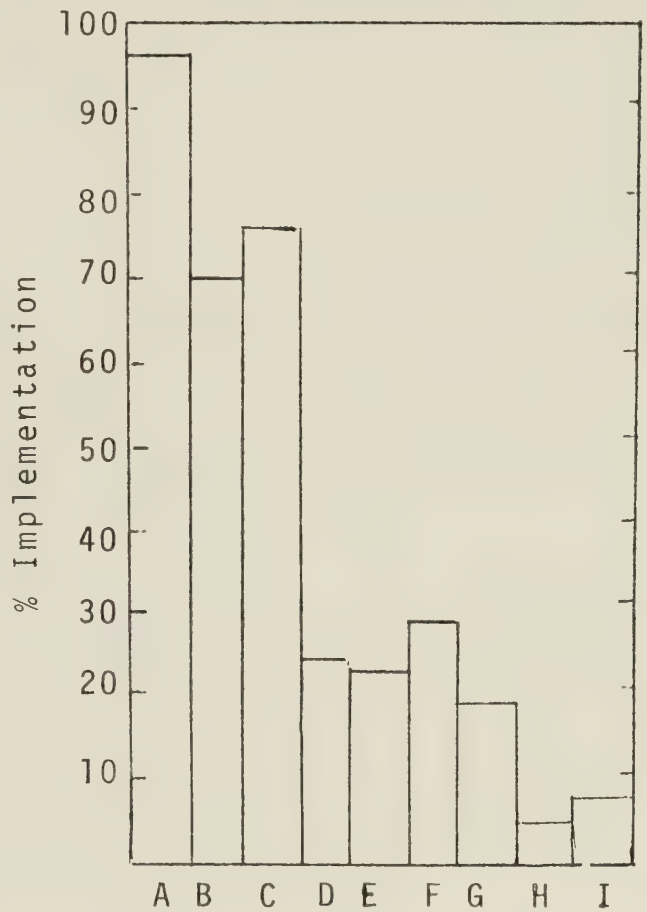
The Israel school system is a centralized one. The official curriculum is common to the whole school system. Thus the 1972-3 curriculum stipulated that geometry should form 25% of the mathematics curriculum. In principal this stipulation should have been implemented in the schools all over the country. As can be seen from Figures 4 to 7 very little of the geometry included in this curriculum is in fact taught in the classroom.

It is interesting to note that where suitable material is available in Hebrew and where there has been some in-service training, the teachers are willing to teach new topics. For example one quarter of the first-grade classes

Figure 4

The Implementation of the Grade 1 Geometry Curriculum in Israel

248 Schools



A. Counting up to ten.

B. Counting in tens up to 50.

C. Intuitive understanding of the rectangle, square, triangle and circle.

D. Intuitive understanding of the rectangular prism, cube and cylinger.

E. Symmetry - using a mirror.

F. Symmetry - by folding.

G. Measuring length.

H. Measuring with a metre.

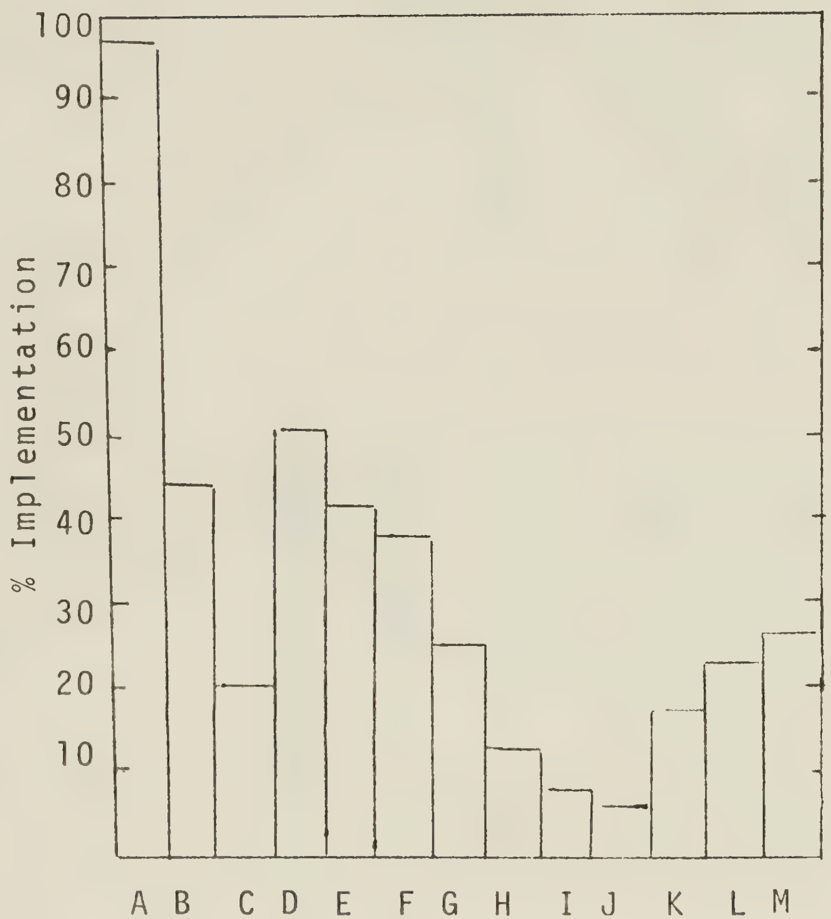
I. Area using a unit area.

(A and B are included for comparisons.)

Figure 5

The Implementation of the Grade 2 Geometry Curriculum in Israeli Schools

240 Schools



A. Understanding of the numbers up to 100.

B. Recognition of shapes in the plane (vertices, sides, diagonals, and the centre of the circle).

C. Recognizing solids - box, cube, cylinder, sphere, cone and pyramid.

D. Intuitive understanding of the straight line and its properties.

E. Recognition of parallel lines.

F. Recognition of the right angle.

G. Symmetry - axes of symmetry of a circle the square and the rectangle.

H. Rotation about a point.

I. Reflection in a plane.

J. Translation - the three transformations.

K. Measurement of length using an agreed unit and parts of the metre.

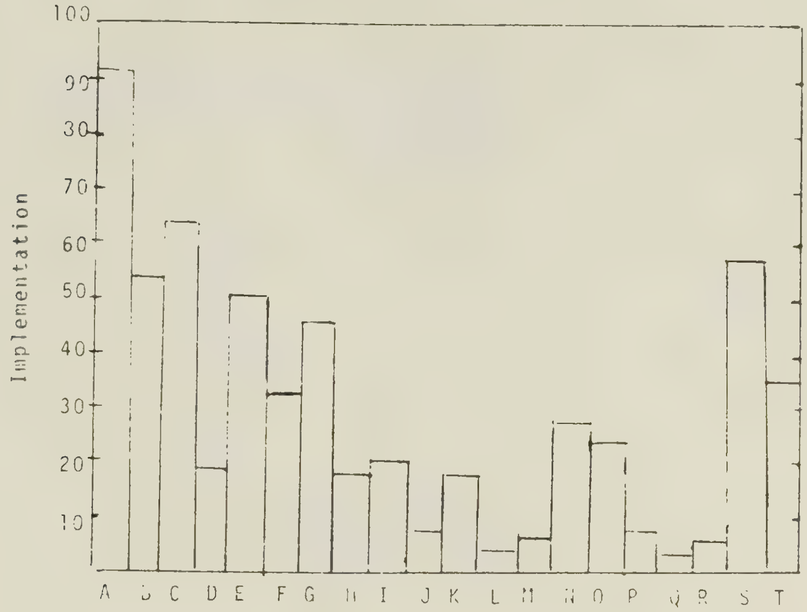
L. Measurement of area using an agreed unit.

M. Measurement of area of a rectangle by counting squares.

Figure 6

The Implementation of the Grade 3 Geometry Curriculum in Israeli Schools

208 Schools



A. Understanding the multiplication table to nines.

B. Drawing using a ruler and a triangle.

C. The straight line, the ray and the segment.

D. Addition and subtraction of segments using a ruler and compasses.

E. Angles, right angles, straight angles, obtuse and acute angles, without measurement.

Classification of triangles by angles and by sides.

G. The properties of the rectangle and the square.

H. Measuring the area of a rectangle and a square by covering with rectangles and squares.

I. The circle, the centre of the circle radius.

J. The ellipse, drawing an ellipse using string.

K. The box, the cube, sphere, cylinder pyramid and wheel.

L. Sections of the cylinder, sphere and wheel.

M. Sections of a box, cone and pyramid.

N. Completion of half of a symmetrical shape relative to a given axis.

O. Finding the axis of symmetry of a given shape (fold line).

P. Translation: the difference between translation and rotation.

Q. Rotational symmetry, covering by rotation, discovering the properties of symmetry.

R. Reflection in a plane. Finding the planes of symmetry of different bodies.

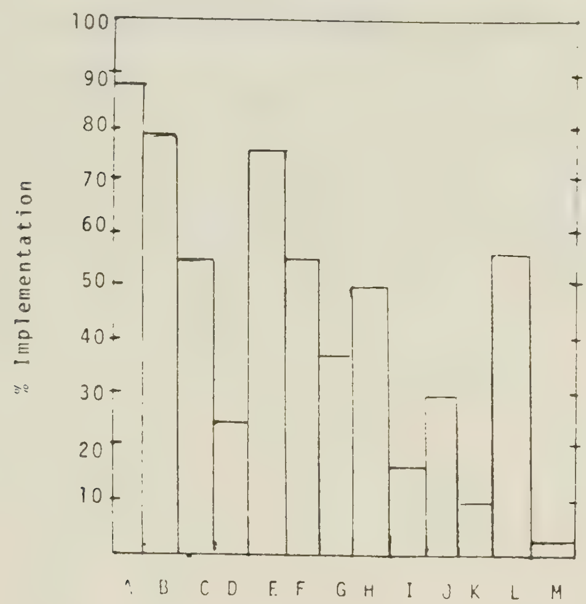
S. Metre, decimetre, centimetre and the millimetre and the relationship between them.

T. Recognition of the basic units of area (cm^2, dm^2, m^2) without changing from one unit to another.

Figure 7

The Implementation of the Grade 4 Geometry Curriculum in Israeli Schools

82 Schools



A. Long division, using distributive law.

B. Parallel and perpendicular lines.

C. Drawing parallel and perpendicular lines using two triangles.

D. The measurement and construction of the heights of different triangles.

E. The family of parallelograms and its properties (the parallelogram, the rectangle, the rhombus and the square).

F. Angles and their measurement using a protractor.

G. Constructions: the use of compasses to draw circles, arcs, isosceles and equilateral triangles.

H. Construction of the circumcircle to the rectangle and the square.

I. Construction of a triangle: (a) given three sides, (b) given a side and two angles and (c) given two sides and the included angles.

J. Demonstration of the properties of a set of parallelograms using their symmetries (axes of symmetry and centre of symmetry).

K. The relation between two reflections and translation.

L. Finding the area of a rectangle by using the formula.

M. Measurement of volume of a rectangular box and the cube using a cubic centimetre.

in the survey are taught symmetry, either using a mirror or by folding paper. The activity-oriented approaches are those of the "Mirror Cards" (Walter, 1974), in Hebrew translation, and "Dafdefet Nikoov" (Translation: Pages for punching holes, Tal, 1975). Other branches of motion geometry, such as rotation and translation are neglected, although they are in the curriculum. This reflects the lack of corresponding materials.

From this brief survey we see that reforms in geometry education, even where they have been adopted, have frequently not been implemented.

Chapter III

The Measuring Instrument

The present study makes extensive use of geometrical sectioning tests to be described in detail in Chapter 4. In the present chapter the evolution of these tests will be reviewed and their reliability will be discussed.

3.1 Sectioning Tests

3.1.1 The Piagetian Tests

Geometrical tests of the type used in this study were described by Piaget and Inhelder (1963). In these tests the child is shown geometric solids and asked to predict the plane shape obtained when the solid is cut by a plane. Two types of response are elicited: in the first the child draws the shape he thinks will appear (the drawing test), and then he chooses the correct drawing from a set of drawings (the multiple-choice test). Piaget maintains that the two tests are equivalent.

In the studies of Piaget and Inhelder, children of 4 to 12 years of age were tested. Among the solids they used were a cylinder, a prism, a parallelepiped, a sphere and a cone. In the first four of these solids the sections seemed to have been transverse and longitudinal. For the cone, the sections produced a circle, a triangle, an ellipse and a parabola. In addition Piaget and Inhelder used some more complex objects, such as an annular ring, a

four-pointed star, a helix and a cornet. However, Piaget and Inhelder do not describe the solids they used in detail, nor do they formulate an exact and consistent protocol. Consequently, it is not always possible to ascertain the precise nature of the cuts which they used.

Piaget and Inhelder found that at the age of four, the child produces drawings which represent a hybrid between the external appearance of the solid and the expected cut. Later, at about 8 or 9 years of age, the child can identify only the simplest sections. By the age of 12 years the child can predict correctly all the sections of the sphere, the cone, the cylinder, the prism, and the parallelepiped, which were presented to them. Piaget thus claims that the child can operate effectively in Euclidean space by the age of 12 years. This age is of interest to us, for most of the children who took part in the present study were in the 6th and 7th grades.

In the tests used in this study the child is confronted with four geometrical solids in the pre-test (a cube, a wholly oblique parallelepiped, a triangular prism, and a right circular cone), and four different solids in the post-test (a rectangular prism, a star-shaped prism, a pyramid and a cylinder). As in the Piagetian study, the child is given a drawing test and a multiple-choice test, as described above. For each body four different sections are presented (Section 4.6 and Appendices 2 and 3). Thus if the Piagetian predictions are valid, the children should

be able to correctly identify all the sections presented in the tests.

3.1.2 Studies Based on the Work of Piaget and Inhelder

Lovell (1959) and Dodwell (1963) both repeated Piaget's work, using similar sectioning tasks. They confirmed the Piagetian conclusions that the child passes through successive levels of maturity in geometric space. However they both find that the age of attainment of the Piagetian stages is quite variable. They also find that maturity in Euclidean space is reached later than was expected by Piaget. This was confirmed by further experiments of Lovell (1965, 1971b).

Rivoire (1961) conducted an investigation of the sequential development of representational space. The children in her study were from 4 years to 15 years of age. Rivoire used a measuring instrument which contained 28 items. She considered four different spaces: projective, topological, affine and Euclidean. For each of the four catagories she used seven different items.

The sequential development of the acquisition of space concepts was confirmed by Rivoire. The child begins to acquire the concepts at about 6 years of age, development continues until a plateau is reached at about 8 years. Further development takes place between 12 and 14 years, when the child reaches full geometrical maturity. She further maintains that even at the age of 14, not all child-

ren have reached full Euclidean maturity. This last statement is at variance with the findings of Piaget and Inhelder.

Boe (1966) also describes a series of tests based on those of Piaget. She also used a drawing and a multiple-choice test. The children were interviewed individually. Four solids were used by Boe:

(1) a cube,

(2) a rectangular prism,

(3) a right circular cylinder,

(4) a right circular cone.

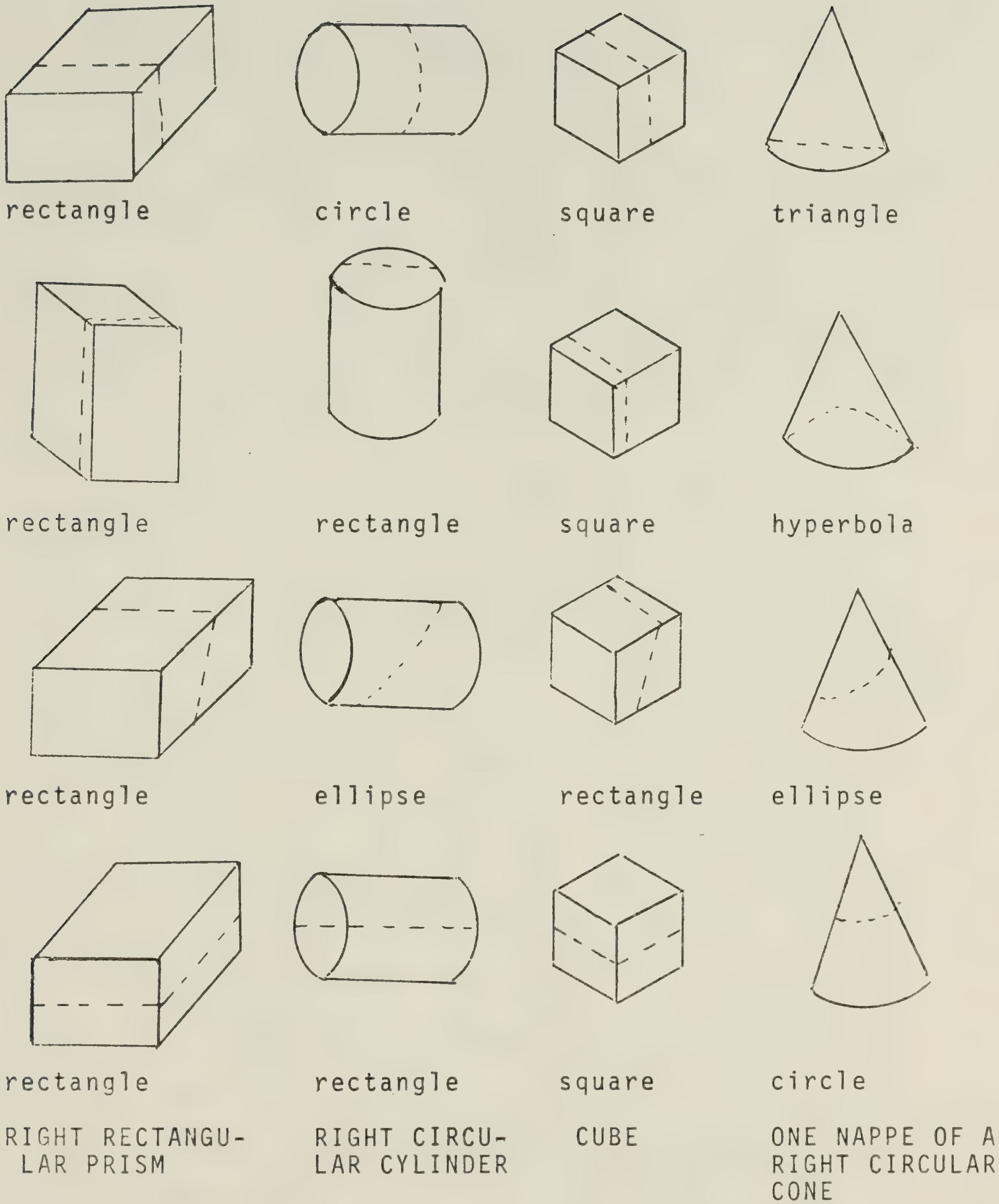
For each of the four solids she presented the following four sections (Figure 8):

(a) a longitudinal cut which was a perpendicular bisector through the major axis,
(b) a transverse cut which was perpendicular, but not a bisector, through the major axis,
(c) an oblique cut which transversed the solid figure oblique to the surface on which it rested, beginning and ending with the bounds of the solid figure,
(d) a parallel cut which was parallel to the surface on which the solid rested.

(Boe, 1966)

Boe found that the order of the solids listed above was the order of increasing difficulty, the cone being the most difficult. Of the sections the oblique cut was more difficult than the other three cuts.

In this study, Boe tested 72 students: 24 students in each of grades 8, 10 and 12. Only 10 of her subjects scored a perfect score on one or other of the tests. These



Dashed lines indicate cuts hypothetically performed. Correct response below each figure.

Figure 8

Sections for Boe's (1966) Tests

students were distributed as follows:

(a) 3 students in grade 8 (12.5%),

(b) 4 students in grade 10 (16.7%),

(c) 3 students in grade 12 (12.5%).

Not one of the students in Boe's sample responded correctly to both forms of the test. Thus she did not agree with Piaget and Inhelder (1963) that children should master all the sections by the age of 12. In addition Boe found that the children of high ability performed better in the tests than did children of lower ability.

A variation of Boe's experiment was performed by Davis (1969). His students were in grades 6, 8 and 10. He used the same solids and the same two methods of response (drawing and multiple-choice) as in Boe's experiment, but paid particular attention to the directions for the presentation of the tests. The children were tested in groups of six and not individually. Davis introduced another element into his study. At the start of the experiment he gave the students an opportunity to acquaint themselves with the sectioning tasks. They looked at irregular solids made of styrofoam and cut the solids with a knife. After this period, the tests were administered.

Davis found that there was some improvement in the scores between grades 6 and 8, but not between grades 8 and 10. This finding is in agreement with that of Boe, that the success of the child in these tests seems to level out and remain constant between grades 8 and 12. However,

Davis's use of the pre-test session is open to criticism. This initial period can be interpreted as a training period for the test to be given later.

The tests used by Boe were incorporated in the work of Bober (1973). He used the tests in conjunction with a training programme based on mathematics laboratory experience in both Euclidean and projective geometry. The students in the study were in grades 7, 8 and 9. Bober used four of the solids used by Boe and Davis: a cube, a rectangular prism, a right circular cylinder and one nappe of a right circular cone. The second and last solids in the list were used in the pre-test. These bodies had been found by Boe to be the easiest and the most difficult for her subjects. Bober also used a triangular prism and a parallelepiped with no rectangular faces, in the pre-test. In the post-test Bober used the remaining two solids from Boe's list (a cube and a right circular cylinder) together with a square-based pyramid and a star-shaped prism. The tests were administered to the class as a whole.

In the present study, the same solids were used as in the work of Bober. However, one slight variation was made: the cube was used in the pre-test and the rectangular prism in the post-test. The reason for this change was that the cube was one of the bodies studied in the unit "Making Rectangular Solids" (Kuper and Walter, 1978), on which this report is based.

Bober studied the relationship between geometrical

maturity and geometrical experience. After the pre-test had been administered, the students were "exposed to mathematics laboratories rich in projective and Euclidean geometric experiences". Bober found that the students had not reached the Euclidean level of maturity. No student obtained a perfect score on the tests. In addition Bober examined the contention of Davis (1970) that it might be more reasonable to accept a score of 24 (75%) as the level required for Euclidean geometrical maturity. Even at this level however, the students had not reached the Euclidean level of maturity. He did find though that there was a significant difference in the scores between those who had the laboratory experience and those who did not. His findings concur with those of Boe and Davis, and clearly emphasize the importance of experience.

Drost (1977) analysed in depth the ability of students to visualize the sections of solids. He also investigated the relationship between this ability and achievement in geometry. His sample was 38 classes of students in grades 5 to 10, and he tested 1004 subjects. The tests were administered to complete classes. Drost found that the ability to visualize the sections of the solids depends on age and on the ability of the student: the higher-ability students performed better than the lower-ability ones. He found that for grades 5 and 8, the sectioning tests were useful in predicting achievement in geometry. There was some evidence that they were useful in grades 7, but this

usefulness was only minimal in grades 9 and 10.

In Drost's study, the present author's improvements in the tests (Chapter IV) were used. Drost also used the present author's protocols for administering the tests, her models, her record sheets and her multiple-choice booklets. The data obtained from the Edmonton part of the study were gathered by Drost and used by him.

One novel feature of Drost's study is that he gives instructions how to interpret the drawing results (Drost, 1977, p. 219). Some of the drawings judged by this author to be squares were not considered squares by Drost. It was for this reason that the author relied on the drawings the children made of known shapes for comparison, and in addition, each drawing was compared with the corresponding choice in the multiple-choice test. These two tests were considered equivalent by Piaget. Thus in the experiment discussed in this report, greater variation of drawing was allowed than was in Drost's study.

The variations, due to the difficulty of interpreting "borderline" drawings, point to the undesirability of testing whole classes of children at one time. In an individualized test, the tester can be quite sure that the child understands what is required of him and at the same time can ask the child to name the shape he is drawing, or ask other questions enabling the tester to ascertain the child's intentions.

3.2 Reliability of the Sectioning Tests

As has been pointed out, no clear details are available of Piaget's method of testing. Boe (1966) calculated the Pearson product-moment correlation coefficient to check the reliability of the two methods of response i.e., the drawing tests and multiple-choice tests. She found the value of this coefficient to be 0.55. This is not a high enough figure to indicate reliability.

The only other study which discussed the reliability of the tests was that of Drost (1977). He found that the Kuder-Richardson 20 coefficients for each of the two tests were 0.81 for the pre-test and 0.91 for the post-test showing that each test was internally consistent. There was a correlation of 0.76 between the pre-test and the post-test. Drost also measured the correlations between the drawing and the multiple-choice responses for each of the sections. These correlations range from 0.1 to 0.55 with a mean of 0.30.

Thus Drost supports Boe's contention that the two methods of response (drawing and multiple-choice) do not measure the same variable.

Drost calculated the correlation coefficient between the two scores on the pre-test and obtained a value of 0.69. For the corresponding results on the post-test the coefficient was 0.81. The correlation coefficient between the total score obtained by the child on the two drawing tests and the total score on the two multiple-choice tests

was 0.85. From these figures Drost concludes that for a set of sectioning tasks the drawing and multiple-choice modes of response are comparable. This result agrees with Piaget and Inhelder's conclusions.

The work reviewed above suggests that the sectioning tests have some value. In the present study, they are used to evaluate the geometrical maturity of the students. However, the author has some misgivings on the methods of presentation. These misgivings will be discussed in Chapter VI.

Chapter IV

The Problem

4.1 The Purpose of the Study

This work will investigate the effect on the geometrical maturity of students of studying a certain unit. The unit is "Making Rectangular Solids" (Kuper and Walter, 1978) described in section 4.5. It was designed to give children experience in manipulating and examining solid bodies before learning algorithms for calculating volume. The concept "Geometrical Maturity" is defined in section 4.2.2.

Specifically an attempt will be made to answer the following question:

Does the study of the unit "Making Rectangular Solids" significantly affect the performance of sixth- and seventh grade children in a geometrical-maturity test?

Two subsidiary questions will be investigated:

(1) Did studying the unit have the same effect on the score in the geometrical-maturity test in Haifa, in Israel as in Edmonton, in Alberta?

(2) Was there any correlation between the IQ of the subject and his scores in the geometrical-maturity test?

4.2 Definitions

4.2.1 Sectioning Tests

A geometrical solid is displayed to the student. He is asked to imagine a knife cutting through the solid in a specified direction from a specified position. In the drawing test, he is asked to draw the outline of the section which would be obtained by the cut. In the multiple-choice test, the student is asked to select the correct section from a group of five drawings (or to reject all five).

4.2.2 Geometrical Maturity

Among the sections used by Piaget and Inhelder (1963) in their studies were the longitudinal and transverse sections of the cylinder, the prism, the parallelepiped, the hollow sphere and the four sections of the cone (giving a triangle, a circle, an ellipse and a parabola. Piaget and Inhelder state that by the age of 12 years children should have mastered these sections.

In the present study, if a child can identify correctly all the 16 sections in the pretest or the post test, we will describe him as being at a level of geometric maturity in Euclidean space.

4.2.3 Geometrical-Maturity Test

These are the sectioning tests described by Boe (1966) and are an adaptation of the tests described by Piaget and Inhelder (1963). The tests and the testing procedures are

described in full in section 4.6.

4.3 The Design of the Study

The study consisted of six stages:

(a) The pilot study (Section 4.7)

(b) The pre-test (Section 4.6.1)

(c) The teaching unit (Section 4.5)

(d) The post-test (Section 4.6.2)

(e) The achievement test (Section 4.6.3)

(f) Analysis of the results (Chapter V).

The sample used in the study is described in section 4.4.

4.3.1 Delimitations

1. The study was delimited to the particular unit described in section 4.5.

2. The sample was taken from grades 6 and 7 only.

3. The students were from five separate schools in Edmonton, Alberta, and from nine schools in Haifa, Israel. The choice of schools in Haifa was restricted to the schools in which there were mathematics coordinators. These teachers had attended courses given by the author of this report as part of their in-service training. As there were more volunteers than were needed, the schools were chosen to give a balance in socio-economic backround of the students.

4. The study was delimited to the particular solids

used in the tests described in section 4.6.

4.3.2 Limitations

1. The study was dependent on the teachers teaching the material in the way intended by the author. The author was not present during the teaching in Edmonton, and therefore has no knowledge of the way the material was taught. The situation was different in Haifa. The teachers were in constant contact with the experimenter, both at regular in-service training sessions and in personal discussions. Every class was visited at least twice during the teaching period. The children were not strangers to the experimenter, as the testing had all been carried out by her.

However, in spite of these precautions there were inevitably differences in approach, as well as in the teaching style of the teachers. These differences might have influenced the effectiveness of the material.

2. It was recommended that the unit be taught in eight to ten lessons. In practice however the time varied between eight and fifteen lessons. Teachers differed in the amount of time they devoted to various sections. In some cases the children became involved in a particular problem and thus the time spent on that section was longer. For example, the experimenter was present at the Degania school when the children sorted the boxes into sets (Appendix 1, page 139). This sorting was intended to be one of four activities for that lesson. In the particular class,

the children had worked on the problem, as a group activity, at the end of the preceding lesson. Each group had recorded its method of sorting the boxes. It took the whole of the next period for each group in turn to demonstrate its classification, whilst the rest of the class guessed the rule for the classification.

3. The variability in teaching time in the different classes was a limitation of the study. One expects that teaching the material more slowly may have a bigger effect on the performance of the children in the tests. However the design of the study did not permit isolation of the time factor. The experimenter was not able to control this factor. In some cases the material was taught at a rate of five lessons a week for two weeks. At the other extreme it was also taught, at one lesson a week over ten to twelve weeks. It is not obvious, *a priori*, which presentation will have the bigger effect. On the one hand, slower exposure to the material may give more time for the concepts to be assimilated. But on the other hand, the infrequent lessons are a break from routine which may be a distraction.

4. Some of the work was done in groups of three or four children. This method of study has both advantages and disadvantages. The group discussions and interactions of the children must be balanced against the possibility that some children might be content to be passive, and thus not be influenced by the material.

5. There was no satisfactory way of ensuring that all the students had been equally exposed to the material. Subjects who were absent from one of the tests were removed from the sample. Students for whom no IQ score was available were also excluded. However the experimenter had no information on the attendance of the remaining children during the teaching period.

6. The testing was performed on the class as a whole. The author does not consider this method of testing to be as reliable as an individual test. This point is discussed more fully in Chapter VI.

4.4 Composition of the Sample

4.4.1 The Canadian Sample

The experiment was performed in Canada in the autumn of 1975 and the spring of 1976. The material was taught in three grade-7 classes in Edmonton. Forty-one children from these classes formed the experimental group. The mathematics-option class at the H.E. Bariault School (designated Class R) was in the experimental group. The other classes in this group were Grade 7b from St Pius (Class S) and Grade 7 from St Vincents (Class T). The distribution of students is shown in Table 5.

The classes which acted as the control group in Edmonton were the remainder of the students in the seventh grade at H.E. Bariault School; i.e. those not in the mathematics-option class (Class X), and Grade 7a from St Pius

Table 5

Distribution of Subjects

	SCHOOL	GRADE	NO. OF SUBJECTS
Haifa: Experimental Group			
A	Fichman	6	17
B	Gordon	6a	23
C	Degania	6	21
D	Sprinzak	6b	18
E	Nirim	6a	15
F	Shalva	6a	29
G	Tel Hai	6b	16
H	Israelia	6b	29
P	Romema	6b	36
		Total	204
Haifa: Control Group			
I	Romema	6a	33
K	Sprinzak	6a	13
L	Nirim	6b	13
M	Shalva	6b	24
N	Tel Hai	6a	22
O	Israelia	6a	28
Q	Romema	6c	29
		Total	162
Edmonton: Experimental Group			
R	H.E. Bariault	7*	14
S	St Pius	7b	13
T	St Vincents	7	14
		Total	41
Edmonton: Control Group			
X	H.E. Bariault	7*	20
Y	St Pius	7a	14
		Total	34

*The mathematics-option class formed the experimental group and the rest of grade-7 were the controls.

(Class Z).

4.4.2 The Israel Sample

The experiment was repeated in Haifa in Israel. It was started at the end of 1976, about one year after the start of the Edmonton experiment. This time lag was due to the fact that all the material had to be translated into Hebrew and printed again. As the material was to form part of the Israel Ministry of Education's curriculum development project, a large sample of students was available.

Haifa is a Mediterranean seaport built on the slopes of Mount Carmel. In general the more prosperous members of the community live on the upper slopes of the mountain, while the working-class areas are along the lower slopes of the mountain and in the bay-side areas to the north. The Jewish population consists mainly of two cultural groups: the Ashkenazim, who originated in Europe and America, and the Sephardim, who reached Israel from the Muslim countries of North Africa and the Middle East. On the whole the Eastern Jews form the culturally-deprived section of the community.

No Arab schools have been included in this sample. However a research student at the University of Haifa has translated the material into Arabic. A parallel study was performed by him in two Arab villages in the Galilee in the spring of 1978.

Nine sixth-grade classes (204 children) formed the experimental group. Another seven sixth-grade classes (162 children) formed the control group. The allocation of classes was made in one of two ways. In schools where there were two or three parallel classes, the selection was made at random. The classes not chosen to be in the experimental group formed the control group. However, after the end of the experiment these classes also studied the material, as it formed part of the sixth-year curriculum.

The control class in the Gordon school was excluded from the final analysis. All the testing had been performed, but no IQ scores were available for this class. In two small schools (Fichman and Degania) there were no parallel grade-six classes to form the control group. However, there were three grade-six classes in the Romema school, two of them being in the control group.

Of the nine schools which participated in the experiment in Haifa, four of them (Fichman, Israelia, Tel Hai, and Romema) are situated on the upper slopes of Mount Carmel, and serve a predominately middle-class population. In two of these schools (Fichman and Tel Hai) mathematics is usually taught in "streamed" classes, i.e., the classes are sub-divided for mathematics lessons into ability groups. In both these schools, the streaming was abandoned while the material in the unit was being taught. The class was treated as a single heterogeneous group. In both these

classes, the home-room teacher taught the material to her class.

Two of the schools, Gordon and Degania, are situated in Kiriat Haim, a working-class suburb in the northern bayside area. Nirim school is in a working-class area close to the southern beach. Most of the population in this suburb are of Sephardic or Eastern Jewish origin. The school is classified as "culturally disadvantaged". It participates in a programme of remedial education, for which it receives special government funds. The last two schools participating in the experiment, Shalva and Sprinzak, are situated on the lower slopes of the Carmel. This is an area of mixed middle-class and working-class populations.

4.5 The Teaching Unit

The teaching unit used in the experiment was the booklet entitled "Making Rectangular Solids". It was written during a workshop at the Curriculum Centre of the Israel Ministry of Education and Culture in the summer of 1974. It has been published in Hebrew as a set of work cards (Kuper and Walter, 1978).

The unit is intended to serve as an introduction to the study of three-dimensional shapes and of volume. These concepts are introduced to the students through a series of "discovery" exercises and games. The unit guides the student through exercises in which he builds a set of rectangular prisms (boxes) from a restricted set of rectangles.

Throughout the work, the child is encouraged to look closely at his solids, to examine them and to play with them. He draws the boxes and compares them with both perspective drawings and with scaled drawings viewed from different angles. Some of the problems studied also introduce simple combinatoric concepts. In parts of the unit the students work individually, while in other parts they work in groups.

4.5.1 Description of the Unit

In the first lessons the students are given a number of pieces of drinking straws of lengths 3 cm, 5 cm, and 8 cm. They are asked to find all the rectangles which can be built with these lengths as side lengths. Six rectangles are possible, of which three are squares.

Next, each group of three or four students are given about 15 rectangles of each of the six types (3 cm by 3 cm; 5 cm by 5 cm; 8 cm by 8 cm; 3 cm by 5 cm; 3 cm by 8 cm; 5 cm by 8 cm). The students are asked to build all possible distinct boxes from the cardboard rectangles. Ten different rectangular prisms can be made from these particular rectangles. Some of the students were able to generalize the problem to the case where n different lengths were specified (Kuper and Walters, 1976).

It is difficult for children in the sixth or seventh grade to calculate the number of boxes it is possible to make. The students have, therefore, to experiment and to

discuss the problem with others in the group, and each has to study carefully his collection of boxes.

At the end of the first section of the unit, the volume of the rectangular box is found by experiment (using centimetre cube blocks), by measurement, and finally - after the formula has been found - by calculation.

In the second section of the unit (Appendix 1 , p. 142) the interest is concentrated on one particular body: the cube. The student has to investigate the 12 pentomino shapes[1]. He then has to sort the shapes into two classes: those which can be folded into open cubes and those which cannot. This classification is completed before the child checks his results by folding the cut-out shapes. This activity gives the student experience in visualizing the change from a two-dimensional shape to a three-dimensional one.

In the third section (Appendix 1, p.146) the student builds an open cube and then studies it. He looks at it from different directions and angles, and compares his box with both full-size and scale drawings of the box. The problem of the number of ways the box could be coloured using five different colours was also investigated.

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[1]A pentomino is a figure made from five congruent squares. The squares are joined so that each square shares a whole side with its neighbor. If n squares are joined in this way to form a shape, the resulting shape is called an n-omino (Golomb, 1965).

In the last section (Appendix 1, p.148), the student finds the relationship between the length of the side of the face diagonal and of the main body diagonal of the cube. He has to find the diagonal of the base and the main body diagonal by experiment. (The question asked for this last case is: "What is the longest stick which will just go into the box?".) In addition he has to find the relationship between these lines and the largest rectangle which can fit into the cube. The number of ways the rectangle can fit into the cube is also explored.

All these exercises are designed to give the child experience in visualizing and looking at special lines and planes in space. The problems involving the longest lines and rectangles are also intuitive solutions of a maximum problem. It is unusual for students to meet this type of problem before studying calculus in the senior high school or in the university.

As an additional exercise, the students in Canada built the five Platonic solids: the regular tetrahedron, hexahedron (cube), octahedron, icosahedron, and dodecahedron. This material was not included in the Haifa experiment. It is being included in other units in preparation by the curriculum centre, which deal with regular and irregular solids, in particular with the Platonic solids, with prisms and with pyramids.

4.6 The Measuring Instrument

Two types of tests were administered to the students: the geometrical-maturity tests and an achievement test. However, difficulties were experienced in the administration of the achievement test, and it was not used in the final analysis of the results.

4.6.1 The Geometrical-Maturity Tests

The geometrical-maturity test in the form used in this experiment was developed by Boe (1966) and adapted by Bober (1973). Further work on this test has been studied by Drost (1977). These studies are described in Chapter 3. The tests are all based on the sectioning of solid figures described by Piaget and Inhelder (1963).

A solid is exhibited to the child. He is asked to predict and draw the shape of the plane figure obtained by cutting the solid as indicated by the tester. A selection of four solids is presented in the pre-test.

After the drawing test has been completed, the same four solids are presented in a different (random) order. For each of the cuts, the child is asked to choose one of five drawings in a multiple-choice test (or, as a sixth option, to reject all five drawings).

The post-test is administered in the same way as the pre-test, but uses a new set of four solids, none of which was used in the pre-test.

4.6.2 The Geometrical-Maturity Test: Pre-test

First the procedure is demonstrated to the class. The demonstration models are prepared so that they can be separated along the plane of the cut. The tester demonstrates the path of the knife through the solid and asks the students to predict the shape of the plane surface formed when the model is opened. Having seen the separated solid and agreed on the shape obtained, the students draw the shape. This procedure ensures that the student knows what is expected of them. In addition it provides four drawings by each student of shapes known both to the students and to the tester. These drawings are helpful when interpreting the drawings made in the test itself. The protocol for the administration of the pre-test is given in Appendix 6.

The solids used in the preparatory (demonstration) stage are a sphere and an octahedron. They provide the following cuts (Fig. 9):

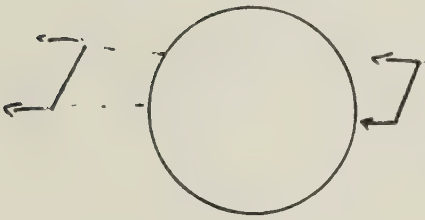
(a) A circle - the sphere cut through the centre.

(b) A square - the octahedron cut through the edges, so that two congruent square-based pyramids were obtained.

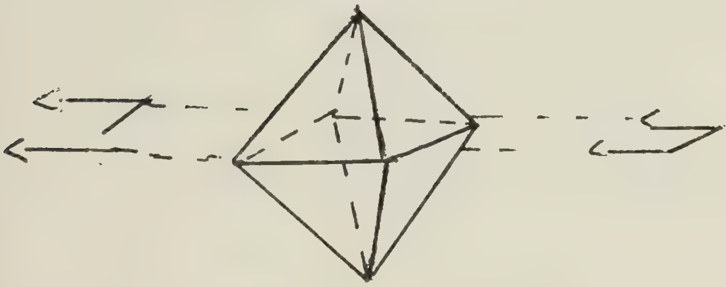
(c) A rhombus - the octahedron cut from vertex to vertex.

(d) A kite - the octahedron cut from the centre of one edge to a point about 1/4 of the side length from the neighbouring vertex (diagonal cut).

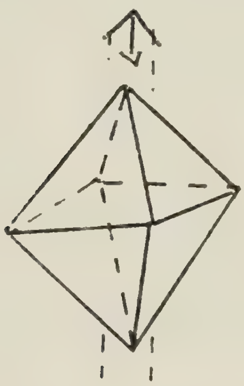
Solid	Section
(a) A sphere	A circle

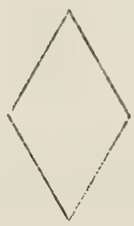


(b) An octahedron	A square

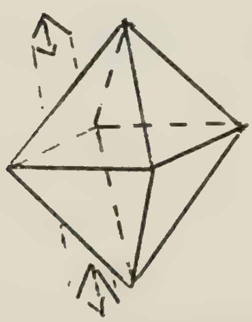


(c) An octahedron	A rhombus





(d) An octahedron	A kite



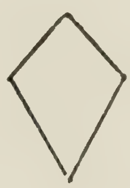


Figure 9

The Demonstration Solids and Sections

The test solids are then shown to the students. For each of the four solids in the pre-test, there are four different sections, i.e. sixteen items in all. The order in which the cuts are presented was first randomized and then presented without change. The same order was used in all the classes tested. When not in use the solids are kept out of sight so that only one solid is visible at a time. The direction of the cut, and the positions of entry and exit of the knife, are carefully demonstrated. The solid is held so that its base is parallel to the floor and the axis vertical.

The models used in the pre-test are:

(a) a cube,

(b) a triangular prism,

(c) a parallelepiped whose faces were all non-rectangular parallelograms, and

(d) a cone.

The directions of the cuts were:

(a) Longitudinal A - through the centre of the solid, perpendicular both to the floor and to the body of the tester,

(b) Longitudinal B - through the centre of the solid, perpendicular to the floor and parallel to the body of the tester,

(c) Transverse - through the centre of the body and parallel to the floor,

(d) Oblique - from the top right of the solid to the bottom left of it.

The only exception to this classification is in the case of the cut longitudinal *B*, for the cone. This cut is made parallel to the axis of the cone but to one side of it. This change was also made by Bober (1973) and Drost (1977). The correct appearance of the cuts, together with a diagram showing the method of cutting, is shown in Figures 10 and 11.

The drawings are collected from the students and the second section of the test is presented. For each cut the child is shown five drawings. He can choose one of the drawings as being the correct representation of the cut or indicate that none of the drawings is appropriate. The drawings, with one modification, are taken from those used by Boe (1966) and Bober (1973).

The order of presentation of the cuts is shown in Figures 10 and 11. The drawings given to the students in the multiple-choice test are shown in Appendix II. In Edmonton the student was given a replica of the pages shown in Appendix 2. He had to circle the letter corresponding to the drawing he thought appropriate or write F (None of the drawings were appropriate). In Haifa each set of five drawings formed one page of a booklet. The same booklets were used in every class. The students recorded their responses by circling the appropriate letter on an answer sheet. A copy of this sheet with an English trans-

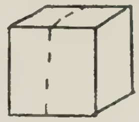
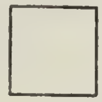
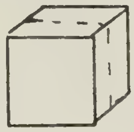
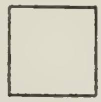
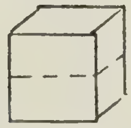
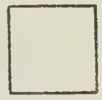
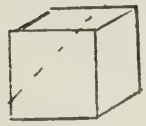
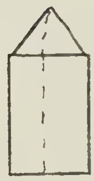
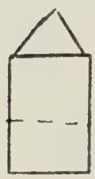
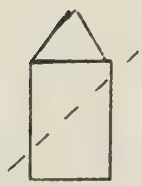
Description	Drawing	Multiple Choice	Direction of Section	Shape of Section
Cube: Longitudinal A	3	8		
Cube: Longitudinal B	12	4		
Cube: Transverse	8	11		
Cube: Oblique	14	16		
Prism: Longitudinal A	15	1		
Prism: Longitudinal B	2	9		
Prism: Transverse	5	15		
Prism: Oblique	9	5		

Figure 10

Order of Presentation and Drawings

of the Sections in the Pre-test

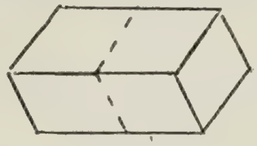
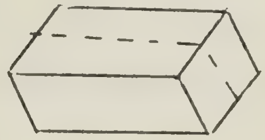
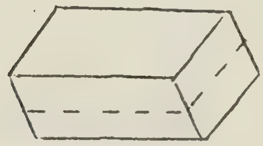
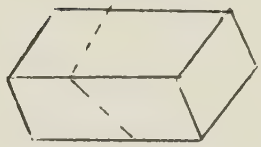
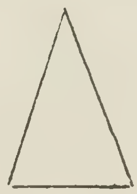
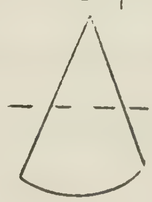
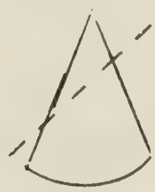
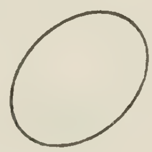
Description	Drawing	Multiple Choice	Direction of Section	Shape of Section
Parallelepiped: Longitudinal A	1	14		
Parallelepiped: Longitudinal B	6	7		
Parallelepiped: Transverse	13	3		
Parallelepiped: Oblique	10	12		
Cone: Longitudinal A	7	6		
Cone: Longitudinal B	16	10		
Cone: Transverse	11	2		
Cone: Oblique	4	13		

Figure 11

Order of Presentation and Drawings

of the Sections in the Pre-test (continued)

lation is included in Appendix IV.

The drawings are scored as follows:

For a correct response: 1.

For an incorrect response: 0.

The scores for all the tests, pre-tests, post-tests and achievement tests are recorded on a record sheet, as shown in Appendix 5.

4.6.3 The Geometrical-Maturity Test: Post-test

The procedures of the post-test are identical to those of the pre-test. The same format for scoring is used. The protocol for administering the test is given in Appendix 7.

The solids used in the post-test were:

(a) a rectangular parallelepiped,

(b) a cylinder

(c) a star-shaped prism, and

(d) a square-based pyramid.

The order of the cuts, both for the drawing test and the multiple-choice test, as well as the drawings and the directions of the cuts in their order of presentation, are shown in Figures 12 and 13. The drawings used for the multiple-choice test are shown in Appendix 3.

When checking the drawings each drawing is examined and the result recorded. After the results are sorted, the corresponding response in the drawing test is compared with that of the multiple-choice test. Wherever there is an inconsistency between the two results, the drawings are

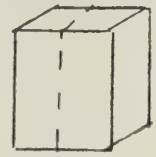
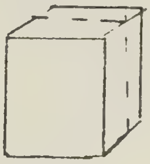
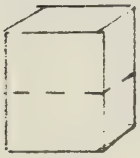
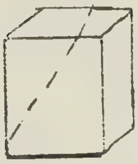
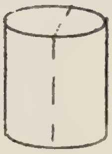
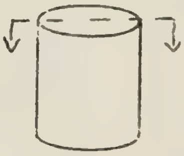
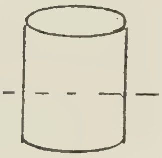
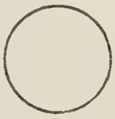
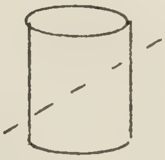
Description	Drawing	Multiple Choice	Direction of Section	Shape of Section
Rectangular box: Longitudinal A	2	5		
Rectangular box: Longitudinal B	7	16		
Rectangular box: Transverse	11	10		
Rectangular box: Oblique	16	3		
Cylinder: Longitudinal A	14	15		
Cylinder: Longitudinal B	12	2		
Cylinder: Transverse	1	12		
Cylinder: Oblique	5	7		

Figure 12

Order of Presentation and Drawings

of the Sections in the Post-test

Description	Drawing	Multiple Choice	Direction of Section	Shape of Section
Star: Longitudinal A	15	4		
Star: Longitudinal B	3	9		
Star: Transverse	6	14		
Star: Oblique	9	8		
Square pyramid: Longitudinal A	4	11		
Square pyramid: Longitudinal B	13	6		
Square pyramid: Transverse	10	13		
Square pyramid: Oblique	8	1		

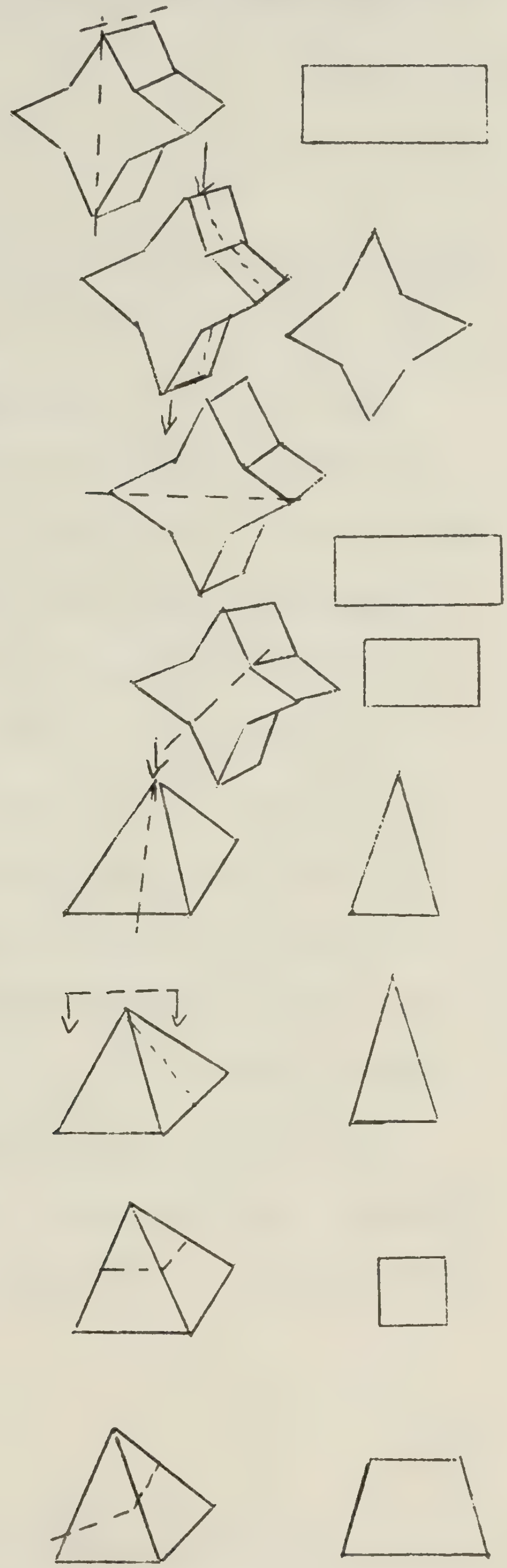


Figure 13

Order of Presentation and Drawings

of the Sections in the Post-test (continued)

reexamined to check against possible misinterpretation. The reliability and validity of the tests are discussed in Chapter III.

4.6.4 The Achievement Test

This test was given within a month after the post-test described above. The tests were not identical in Haifa and in Edmonton. The changes introduced in the Haifa achievement test were necessitated by the fact that the material was being tested for the Curriculum Centre of the Ministry of Education. Many of the questions included in the tests were designed to test if the material had been learned to a satisfactory level. These questions were directly concerned with recall of material in the unit.

Two of the control classes were given this test. After the test they began to study the unit. However, in the other control classes, the teachers began to use the unit *after* the administration of the post-test described in section 4.6.3, but *before* the administration of the achievement test. These tests were therefore not included in the final analysis. The tests themselves are included in Appendices 8 and 9.

4.7 The Pilot Study

A pilot study was conducted in Edmonton in April and May of 1975. The purpose of the pilot study was to see if the material needed refinement and to test the method of

applying the tests.

Two classes were used: a grade-six class from the Talmud Torah school, and a grade-six class from the West Edmonton Christian School. Both these schools are private denominational schools, and most of the children attending them are from middle-class backgrounds. The author taught the unit at the Talmud Torah School, while the class teacher taught the material at the West Edmonton Christian School. The testing was conducted in both schools by the author of this report.

The unit was taught in the Talmud Torah School for three periods of an hour each week for three weeks. The teaching at the West Edmonton Christian School lasted about a month. The pre-test, the post-test and an achievement test were administered. As a result of the pilot study some changes were made in the material and the tests.

After the material was translated into Hebrew, the material was tested in a further pilot study in six schools in Haifa in May and June 1976. Only the achievement test was given on this occasion.

4.7.1 Improvements in the Unit

The only major change made in the unit in the light of the pilot study in Edmonton was the decision to use unit blocks (or Cuisenaire rods) for measuring volume. It had originally been anticipated that the child would be free to choose any suitable solid material (sand, beans, etc.) for

comparing the volumes. However, the use of blocks, whose edge lengths were whole numbers of centimeters, proved an efficient method of introducing the algorithm for measuring volume. Indeed, the use of the blocks enabled many of the children to discover the algorithm for themselves.

In Israel an additional page was added clarifying the meanings of the terms: edge, side, face and vertex. One other precaution was taken in the light of experience in the pilot study in Israel. The cardboard rectangles were supplied by the workshop of the Curriculum Centre in Jerusalem. On his own initiative the technician supplied each type of rectangle cut from a different coloured card. At one in-service training period, it was discovered that each type of rectangle having its own characteristic colour completely frustrated the purpose of the first section of the unit. Instead of deciding whether every different type of rectangle had been used for building the boxes, the students looked for boxes by the colours of the faces, and ignored the differences in size. Thus, it is important to ensure that all the rectangles are made from card of the same colour.

In order to minimize the amount of time spent in building boxes, the exercise on page 25 of the Unit (Appendix 1, p. 148) was modified. In order to find the number of different ways an open cube could be coloured using five different colours, the child was provided with a duplicated page containing at least forty nets of the open cube, as in

the diagram below (Fig. 14).

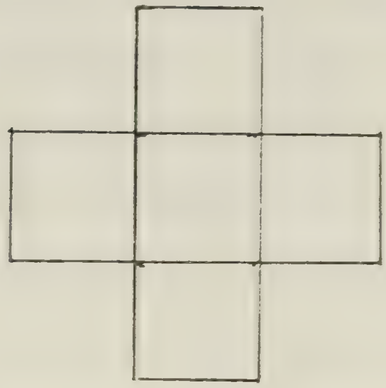


Figure 14

Net of Open Box

The students coloured these diagrams and not the boxes. However a set of 30 boxes showing all possible arrangements was available when the class discussed the problem.

4.7.2 Improvements in the Tests: The Sectioning Tests

The sectioning tests were used as shown in Bober (1973). In the light of experience gained in the pilot study, the protocols provided for the tester were clarified. In particular, the method of holding the solid during the demonstration of the cuts was carefully described. In the pilot study the multiple-choice test of Bober was used. Afterwards some changes were made in the distractors.

Bober had based his work on the study of Boe (1966). The distractors were taken from the incorrect drawings of children from first to twelfth grade. In the present study the distractors were redrawn to scale. The only other

change of substance was in the drawings for the oblique cut of the triangular prism (Appendix 2, p. 156, No. 5). It was observed that for this section children quite often drew an equilateral triangle. However, in the multiple-choice test, Bober presented only *one* triangle, namely the correct iscoceles, but not equilateral, triangle.

Thus in the final version of the multiple-choice test two triangles were included for the oblique cut of the triangular prism: an equilateral triangle and an iscoceles, but not equilateral, triangle.

4.7.3 Improvements in the Tests: The Achievement Test

The version of the achievement test used in the pilot study proved too long and difficult. The achievement test as finally used in Edmonton is shown in Appendix 9. The rejected questions are shown at the end of this test.

The test designed for the Haifa pilot study was different from that used in Edmonton. As the material was being evaluated for the Ministry of Education's Curriculum Development centre, the test used in Edmonton was not suitable. An English translation of the Haifa test is shown in Appendix 8.

4.8 The Teachers and the Teaching of the Unit

After a short training period, the teachers in Canada were left mainly on their own. However, they did receive a teacher's guide which was detailed and comprehensive (Appendix 10). As no visits were made to the classroom while the unit was being taught, no comment can be made on the methods of teaching used. It was stated by the supervisor of the experiment that the instructions left in Edmonton by the author were followed reasonably well. However there is no complete assurance that this was so.

In Haifa, the teachers taking part in the experiment were all trained mathematics coordinators. They normally taught methematics to the class participating in the experiment. The exceptions occurred where mathematics was taught in streamed-ability groups. The streaming was temporarily suspended, so that all the classes in Haifa were heterogeneous. The teachers taking part in the experiemt were given detailed in-service training. Regular visits were made to see the class at work.

The pupils were tested about one month before starting the unit. In Haifa they were tested by the author, and in Edmonton by a research student at the University of Alberta. In all but one of the schools, the work was completed in eight to twelve lessons spread over six to twelve weeks. The exception was at the Sprinzak School in Haifa, where most of the material was taught in the course of two weeks in May.

All the children in Haifa had studied area, but had not learned volume. There was no information reported to the author on the method of instruction in Edmonton, but it has been assumed that it was similar to that in Haifa.

Chapter V

Analysis of the Data

In this chapter the experimental data are examined. In Section 5.1, the composition of the sample is discussed. The mean IQ score for each class is listed and examined for normality.

In Section 5.2, the data of the various tests are analysed. The pre-tests are examined for normality and ANOVA is calculated for each of the Haifa and Edmonton samples, as well as the two groups treated as a single entity. For the post-test the ANOVA is calculated between the experimental and control groups, for Haifa and for Edmonton separately. Section 5.2.1 deals with the drawing tests, Section 5.2.2 with the multiple-choice tests, and Section 5.2.3 with the total scores.

The correlation of IQ score with the test scores is discussed in Section 5.3. The level of geometrical maturity is discussed in Section 5.4. Finally in Section 5.5 the questions posed in Chapter IV will be discussed.

5.1 Description of the Sample

Seventeen grade-6 classes in Haifa and five grade-7 classes in Edmonton participated in the experiment. There were 676 children distributed as follows:

(1) 283 in the experimental group in Haifa,

(2) 252 in the control group in Haifa,

(3) 68 in the experimental group in Edmonton,

(4) 73 in the control group in Edmonton.

However the final sample used in the analysis of the data was much smaller. Some subjects were eliminated because they were absent from one or other of the tests. Others were eliminated because IQ scores were not available. In Haifa, one complete control class (at the Gordon school) was eliminated for this latter reason. The composition of the residual sample was:

(1) 204 in the experimental group in Haifa,

(2) 162 in the control group in Haifa,

(3) 41 in the experimental group in Edmonton,

(4) 34 in the control group in Edmonton,

a total of 441 subjects. The distribution of the students is shown in Table 6.

The average age of the students in Israel was 11 years, 6.4 months, with a standard deviation of 3.9 months. In Edmonton the corresponding figures were 12 years 4.1 months, and 6.0 months. The Haifa children were in grade 6, those in Edmonton were in grade 7. For this reason the final results were analyzed as two parallel experiments rather than pooled as one single experiment.

The mean IQ for each class is also shown in Table 6. The test used in Edmonton is a Lorge-Thorndike Intelligence Test, administered by the school authorities. The results are quoted in percentiles. The "Milta" test, used in

Table 6

Distribution of Subjects and IQ

	SCHOOL	GRADE	NO. OF SUBJECTS	MEAN IQ	S.D.
Haifa: Experimental Group					
A	Fichman	6	17	110.6	8.1
B	Gordon	6a	23	104.6	10.6
C	Degania	6	21	102.4	14.6
D	Sprinzak	6b	18	101.3	9.2
E	Nirim	6a	15	96.9	10.7
F	Shalva	6a	29	107.0	12.6
G	Tel Hai	6b	16	107.1	7.7
H	Israelia	6b	29	115.3	8.2
P	Romema	6b	36	112.4	10.2
		Total	204	107.5	11.8
Haifa: Control Group					
I	Romema	6a	33	112.5	10.9
K	Sprinzak	6a	13	98.8	11.3
L	Nirim	6b	13	100.9	13.1
M	Shalva	6b	24	101.7	12.7
N	Tel Hai	6a	22	111.0	11.2
O	Israelia	6a	28	110.3	9.7
Q	Romema	6c	29	113.7	9.3
		Total	162	108.4	12.2
Edmonton: Experimental Group				(Percentiles)	
R	H.E. Bariault	7*	14	70.9	25.0
S	St Pius	7b	13	42.7	19.0
T	St Vincents	7	14	59.5	31.1
		Total	41	58.1	27.2
Edmonton: Control Group					
X	H.E. Bariault	7*	20	61.1	25.9
Z	St Pius	7a	14	44.2	28.8
		Total	34	54.1	27.6

*See footnote Table 5.

Israel, is similar in content to the Lorge-Thorndike test. This test is standardized to give an average score of 100 in Israel as a whole. However the mean performance in Haifa is usually higher than the national average, as is reflected in the mean score of 107.5 for the experimental group and 108.4 for the control group in this report. In Edmonton the mean percentile figures were 58.1 for the experimental groups and 54.1 for the control group.

The four groups (experimental and control in Haifa, and experimental and control in Edmonton) were treated independently. The distribution of the IQ scores in each group was tested for normality of error distribution. Splitting the errors into four cells yields the frequencies shown in Table 7. In every case the value of χ^2 for $p = 0.05$ and 3 degrees of freedom is less than the critical value of 7.81. Thus the null hypothesis of normality of distribution of errors is not rejected; i.e., the IQ's can be considered as normally distributed.

5.2 Analysis of Data

The drawings and the responses to the multiple-choice test were scored as follows: 1 for a correct response and 0 for an incorrect or missing response. The scores were sorted so that it was possible to compare at a glance the response for a particular section in the drawing test with the corresponding response in the multiple-choice test. If the two scores were different, the drawing for that partic-

Table 7

Error Frequency: IQ

Quartiles	Haifa		Edmonton	
	Experimental	Control	Experimental	Control
$-\infty < e_{ik} \le Q$	41	41	11	9
$-Q < e_{ik} \le 0$	62	35	8	4
$e_{ik} \le Q$	57	46	8	14
$Q < e_{ik} < \infty$	44	40	14	7
Square root σ of the pool variance	11.81	12.23	27.23	27.01
$Q \equiv 0.6745\sigma$	7.97	8.25	18.35	18.02
χ^2 (3)	6.0	1.51	2.4	6.23

Notes: 1. The quartiles are $\pm$ 0.6745σ, where σ is the square root of the pool variance.
2. The critical value of χ^2 (3) is 7.81, p = 0.05.

ular section was checked to ascertain if it had been interpreted incorrectly. Thus if the child chose a square in a multiple-choice selection, and the corresponding drawing was nearly, but not quite, a square, the drawing was taken to be a square.

The maximum score in the drawing and multiple-choice tests was 16 points, making a total of 32 for each of the pre- and post-tests. However all scores given in the following analysis are recorded as a percentage. The mean mark and its standard deviation for each class and for each test, as well as the gain score between pre-test and post-test, are shown in Tables 8 to 10. The results for each of these tests and for each group are summarized in Table 11.

5.2.1 The Drawing Test

The mean value of the scores for the drawing pre-test for the Haifa experimental group was 55.2, and for the Haifa control group was 55.1 (Table 8). The corresponding scores in Edmonton were 53.2 and 53.9 respectively. Analysis of variance between these four groups (Table 12) gave an F value of 0.17 (Table 12). The critical value of $F(3,437)$, $p = 0.05$, is 2.62. Thus the null hypothesis that there is no difference between the four groups, cannot be rejected. Nor is there any significant difference between the results in the drawing pre-test of the two Haifa groups nor the two Edmonton groups. In other words the

Table 8

Results of Drawing Test

Class	Number of Students	Pre-test Mean	Pre-test S.D.	Post-test Mean	Post-test S.D.	Gain Score Mean	Gain Score S.D.
Haifa: Experimental Group							
A	17	64.0	16.6	73.2	15.0	9.2	10.3
B	23	54.1	15.6	69.8	16.7	15.2	20.6
C	21	47.9	18.0	83.0	6.4	35.4	17.8
D	18	39.9	25.7	60.1	19.2	20.1	20.5
E	15	39.2	19.8	53.8	20.4	14.6	19.7
F	29	56.3	20.4	74.6	20.0	18.3	16.8
G	16	60.1	15.7	78.9	11.9	18.4	16.3
H	29	62.7	20.2	77.4	16.4	14.7	13.1
P	36	61.1	19.0	80.6	16.6	19.4	19.1
Total	204	55.2	21.2	73.8	18.6	18.5	18.8
Haifa: Control Group							
I	33	56.6	19.3	72.6	21.0	14.5	20.1
K	13	39.9	18.9	45.7	17.4	5.8	21.0
L	13	44.2	16.0	57.2	17.6	13.0	18.3
M	24	42.4	16.4	49.2	15.1	7.0	14.4
N	22	63.6	10.1	80.4	12.8	16.8	13.9
O	28	67.0	19.6	79.2	19.1	12.3	16.2
Q	29	57.8	22.3	79.5	14.6	21.8	21.5
Total	162	55.1	20.5	69.2	21.7	13.7	17.7
Edmonton: Experimental Group							
R	14	60.7	16.7	74.1	19.7	13.4	13.8
S	13	49.5	14.8	71.6	12.9	22.1	14.1
T	14	49.1	29.1	71.0	23.8	21.9	12.9
Total	41	53.2	21.5	72.3	19.1	19.1	13.9
Edmonton: Control Group							
X	20	51.3	19.4	70.0	22.6	18.8	14.9
Z	14	57.6	22.9	70.5	17.4	12.9	24.6
Total	34	53.9	20.5	70.2	20.0	16.3	19.7

All scores are given as percentages.

Table 9

Results of Multiple-Choice Test

Class	Number of Students	Pre-test Mean	Pre-test S.D.	Post-test Mean	Post-test S.D.	Gain Score Mean	Gain Score S.D.
Haifa: Experimental Group							
A	17	63.6	15.4	73.5	17.5	9.9	21.6
B	23	56.5	16.3	62.2	19.4	5.7	21.8
C	21	58.1	15.7	73.8	14.2	18.8	19.9
D	18	51.0	20.1	55.6	23.3	3.8	16.4
E	15	45.8	15.4	55.8	17.0	10.4	18.1
F	29	64.0	18.4	67.7	21.3	3.9	17.4
G	16	69.5	13.4	79.7	12.0	10.2	14.5
H	29	66.8	19.0	74.4	18.5	7.3	14.1
P	36	70.3	15.8	76.6	15.9	5.9	55.7
Total	204	61.7	18.6	69.7	19.7	8.1	18.3
Haifa: Control Group							
I	33	58.8	19.1	70.6	22.2	11.8	20.9
K	13	48.6	14.1	41.8	20.9	-6.7	22.0
L	13	45.2	20.8	57.2	15.1	12.0	15.2
M	24	52.9	15.0	40.4	17.1	-12.5	16.0
N	22	63.4	12.1	75.0	15.2	11.7	12.5
O	28	68.8	15.3	75.5	13.8	5.8	17.1
Q	29	65.5	18.6	75.2	13.8	9.7	17.3
Total	162	59.8	18.2	64.5	23.8	5.0	19.3
Edmonton: Experimental Group							
R	14	67.9	12.7	65.6	15.6	-2.2	15.4
S	13	62.5	16.1	62.5	14.4	0.0	27.3
T	14	59.4	12.4	62.5	18.8	3.1	16.0
Total	41	63.2	13.7	63.6	15.9	0.3	15.1
Edmonton: Control Group							
X	20	65.9	12.9	57.8	18.1	-8.1	21.3
Z	14	67.6	21.2	60.3	20.9	2.7	15.3
Total	34	62.5	16.8	58.8	18.8	-3.7	19.2

All scores are given as percentages.

Table 10

Results of the Total Score
(Drawing Test and Multiple Choice)

Class	Number of Students	Pre-test Mean	Pre-test S.D.	Post-test Mean	Post-test S.D.	Gain Score Mean	Gain Score S.D.
Haifa: Experimental Group							
A	17	63.8	15.6	73.3	15.3	9.6	14.6
B	23	55.3	15.0	66.0	16.3	10.7	19.2
C	21	51.5	15.2	78.4	8.8	26.9	16.1
D	18	45.5	23.4	57.8	20.9	12.3	15.6
E	15	42.5	16.7	54.8	17.9	12.5	17.2
F	29	60.1	18.2	71.2	19.9	11.1	14.2
G	16	65.0	14.3	79.3	11.5	14.3	15.1
H	29	64.9	18.9	75.9	17.1	11.0	12.3
P	36	65.7	16.4	78.7	16.7	12.8	16.7
Total	204	58.5	18.6	71.8	18.1	13.3	16.1
Haifa: Control Group							
I	33	59.1	16.8	71.2	21.3	12.1	15.6
K	13	44.2	15.1	43.8	18.4	-0.5	20.2
L	13	44.7	16.7	57.2	16.3	12.5	13.3
M	24	47.7	13.9	44.9	15.4	-2.7	13.2
N	22	63.5	10.2	77.7	13.5	14.2	12.0
O	28	67.9	16.6	77.3	18.8	9.5	14.4
Q	29	61.6	19.8	77.4	13.6	15.7	18.2
Total	162	57.6	17.8	67.0	21.5	9.4	16.4
Edmonton: Experimental Group							
R	14	64.3	12.6	69.9	17.0	5.6	12.0
S	13	56.0	14.0	67.1	12.2	11.1	11.3
T	14	54.2	18.7	66.7	20.4	12.5	8.8
Total	41	58.2	15.4	67.9	16.4	9.7	10.8
Edmonton: Control Group							
X	20	58.6	13.9	63.9	18.8	5.3	15.2
Z	14	57.6	19.2	65.4	17.3	7.8	14.5
Total	34	58.2	15.8	64.5	17.6	6.3	14.5

All scores are given as percentages.

Table 11

Summary of Results

		I	II	III	IV
Drawing Pre-test	$\bar{x}$ S.D.	55.2 21.2	55.1 20.5	53.2 21.5	53.9 20.5
Drawing Post-test	$\bar{x}$ S.D.	73.8 18.2	69.2 21.7	72.3 19.1	70.2 20.0
Multiple Choice Pre-test	$\bar{x}$ S.D.	61.7 18.6	59.8 18.2	63.7 13.7	62.5 16.8
Multiple Choice Post-test	$\bar{x}$ S.D.	69.7 19.7	64.5 23.8	63.6 15.9	58.8 18.8
Total Score Pre-test	$\bar{x}$ S.D.	58.5 18.6	57.6 17.8	58.2 15.4	58.2 15.8
Total Score Post-test	$\bar{x}$ S.D.	71.8 18.1	67.0 21.5	67.9 16.4	64.5 17.6
Gain Score Drawing	$\bar{x}$ S.D.	18.5 18.8	13.7 17.7	19.1 13.9	16.3 19.7
Gain Score Multiple Choice	$\bar{x}$ S.D.	8.1 18.3	5.0 19.3	0.3 15.1	-3.7 19.2
Gain Score Total	$\bar{x}$ S.D.	13.3 16.1	9.4 16.4	9.7 10.8	6.3 14.5

All scores are given as percentages.

I :Haifa-Experimental Group

II :Haifa-Control Group

III:Edmonton-Experimental Group

IV :Edmonton-Control Group

Table 12

Analysis of Variance: Drawing Pre-test

Source	df	SS	MS	F-Ratio
All four groups				
Between	3	5.70	1.90	
Within	437	4864.14	11.13	
				0.17*
Haifa: Experimental-Control				
Between	1	0.17	0.17	
Within	364	4025.86	11.06	
				0.02**
Edmonton: Experimental-Control				
Between	1	0.21	0.21	
Within	73	758.27	10.39	
				0.02***

* $F(3,437)$, $p = 0.05$, is 2.62

** $F(1.364)$, $p = 0.05$, is 3.87

*** $F(1, 73)$ $p = 0.05$, is 3.98

initial performance of all four groups was equivalent. (Ferguson, 1959)

The distributions of the drawing pre-test scores were tested for normality. The results of this examination are recorded in Table 13. The null hypothesis that the errors are normally distributed is not rejected. The value of χ^2 for p = 0.05 and 3 degrees of freedom required for rejection of the null hypothesis is greater than 7.81. All the values calculated for χ^2 in Table 13 are less than this value. Therefore it can be assumed that, at the start of the experiment, all the groups were normally distributed with respect to their performance on the drawing test.

In the post-tests the mean score for every group was higher than in the pre-test. The values in Haifa were 73.8 for the experimental group and 69.2 for the control group. The corresponding figures in Edmonton were 72.3 and 70.2 respectively.

Table 14 shows the analysis of variance between pre-test and post-test. In Haifa a value of F = 4.95 was obtained. The critical value of $F(1,364)$, p = 0.05, is 3.87. Thus the null hypothesis that there is no difference between the post-test results of the experimental group compared with the control group, can be rejected. There is a significant difference between the achievement of the two Haifa groups in the drawing post-test.

In Edmonton the calculated value of F is 0.20. The

Table 13

Error Frequency: Drawing Pre-test

Quartiles	Haifa Experimental	Haifa Control	Edmonton Experimental	Edmonton Control
$-\infty < e_{ik} \leq -Q$	51	30	9	9
$-Q < e_{ik} \leq 0$	49	51	12	8
$0 < e_{ik} \leq Q$	53	44	9	6
$Q < e_{ik} < \infty$	51	37	11	11
Square root σ of the pool variance	21.2	17.8	27.2	20.5
$Q \equiv 0.6745\sigma$	14.3	12.0	18.4	13.8
χ^2 (3)	0.16	6.05	0.66	1.53

Notes: 1. The quartiles are $\pm$ 0.6745σ , where σ is the square root of the pool variance.
2. The critical value of χ^2 (3) is 7.81, p = 0.05.

Table 14

Analysis of Variance: Drawing Post-test

Source	df	SS	MS	F-Ratio
Haifa: Experimental-Control				
Between	1	50.57	50.57	
Within	364	3720.65	10.19	
				4.95*
Edmonton: Experimental-Control				
Between	1	1.97	1.97	
Within	73	722.22	9.89	
				0.20**

* $F(1,364)$, p = 0.05, is 3.87

** $F(1, 73)$, p = 0.05, is 3.98

critical value of $F(1,73)$, $p = 0.05$, is 3.98. Thus the calculated value is lower than the value obtained from the tables, and the null hypothesis that there is no difference between the performance of the two groups in the post-test cannot be rejected.

5.2.2 The Multiple-Choice Test

The mean value of the score in the pre-test for the experimental group in Haifa in the multiple-choice test was 61.7 and for the control group the score was 59.8 (Table 9). The corresponding figures for Edmonton were 63.7 and 62.5 respectively. Analysis of variance to test the null hypothesis H_o that there is no difference between the means of the data in the pre-test gives values below 1 (Table 15). This was true for the four groups treated as a single experiment as well as for Haifa and Edmonton separately. H_o is therefore not rejected. Thus we can assume that initially there was no difference between experimental and control groups in their performance in the multiple-choice test.

As in the drawing test, the mean score for each of the experimental groups in the multiple-choice post-test showed an increase over the corresponding score for the control group. Thus the Haifa scores were respectively 69.7 and 64.5 whilst in Edmonton they were 63.6 and 58.8 respectively.

The pre-test distributions for all groups were examined for normality (Table 16). The values of χ^2 obtained for the pre-tests show that there is no deviation from

Table 15

Analysis of Variance: Multiple-Choice Pre-test

Source	df	SS	MS	F-Ratio
All four groups				
Between	3	15.69	5.23	
Within	437	3615.70	8.27	
				0.63*
Haifa: Experimental-Control				
Between	1	8.45	8.45	
Within	364	3171.31	8.71	
				0.97**
Edmonton: Experimental-Control				
Between	1	0.28	0.28	
Within	73	444.39	6.09	
				0.05***

*$F(3,437)$, $p = 0.05$, is 2.62

**$F(1,364)$, $p = 0.05$, is 3.87

***$F(1, 73)$, $p = 0.05$, is 3.98

Table 16

Error Frequency: Multiple-Choice Pre-test

Quartiles	Haifa Experimental	Haifa Control	Edmonton Experimental	Edmonton Control
$-\infty < e_{ik} \leq -Q$	57	36	11	7
$-Q < e_{ik} \leq 0$	45	50	8	10
$0 < e_{ik} \leq Q$	50	35	10	7
$Q < e_{ik} < \infty$	52	41	12	10
Square root σ of the pool variance	18.6	18.2	13.7	16.8
$Q \equiv 0.6745\sigma$	12.6	12.3	13.7	13.3
χ^2 (3)	1.45	3.48	0.85	1.06

Notes: 1. The quartiles are $\pm 0.6745\sigma$, where σ is the square root of the pool variance.
2. The critical value of χ^2 (3) is 7.81, at p = 0.05.

normality. All the values obtained were less than the value $\chi^2 = 7.81$ for three degrees of freedom at $p = 0.05$.

Table 17 shows the analysis of variance for the means in the multiple-choice test. In Haifa, the F value of 5.20 enables us to reject the null hypothesis that there is no difference in the means. The critical value of $F(1,364)$, $p = 0.05$, is 3.84. The value of F for the Edmonton results was 1.39 (critical value $F(1,73)$, $p = 0.05$, for rejection is 4.1). Thus for the Edmonton results, the null hypothesis that there is no difference between the means cannot be rejected.

This analysis shows that after studying the unit the Haifa experimental group performed better in the multiple-choice test than did the control group. In Edmonton, the additional gain in this test of the experimental group, compared with the control group, was not significant.

5.2.3 The Total Score

The analyses described above were performed on the total score obtained by each student. This score was found by adding the scores on the drawing test and the corresponding multiple-choice test. From Table 10 it can be seen that the mean score for the pre-test in Edmonton was the same for each group, i.e. 58.2%. The corresponding scores for the Haifa experiment were 58.5 and 57.6. The analysis-of-variance table, shown in Table 18 gives the F value for all four pre-test groups together, as well as for the pre-

Table 17

Analysis of Variance: Multiple-Choice Post-test

Source	df	SS	MS	F-Ratio
Haifa: Experimental-Control				
Between	1	62.35	62.35	
Within	364	4377.59	11.99	
				5.20*
Edmonton: Experimental-Control				
Between	1	10.71	10.71	
Within	73	572.04	7.73	
				1.39**

*$F(1,364)$, $p = 0.05$, is 3.87

**$F(1, 73)$, $p = 0.05$, is 3.98

Table 18

Analysis of Variance: Total Score Pre-test

Source	df	SS	MS	F-Ratio
All four groups				
Between	3	6.57	2.19	
Within	437	14363.78	32.87	
				0.07*
Haifa: Experimental-Control				
Between	1	6.47	6.47	
Within	364	12496.24	34.33	
				0.19**
Edmonton: Experimental-Control				
This result was not analyzed as there was no difference between the means.				

*$F(1,437)$, p = 0.05, is 2.62

**$F(1,364)$, p = 0.05, is 3.87

test experimental and control groups in Haifa. In both these cases the calculated value of F was less than 1. Thus the null hypothesis that there is no significant difference between the means of these figures cannot be rejected.

The mean scores for the post-tests in Haifa were 71.8 for the experimental group and 67.0 for the control group. The corresponding figures in Edmonton were 67.9 and 64.5 respectively.

The pre-test total-score distributions were tested for normality. As can be seen in Table 19, all the pre-test values of χ^2 are below the tabulated value of 7.81 for the 0.05 value of p and for three degrees of freedom.

The analysis-of-variance calculation for comparison of the experimental and control groups in the total-score post-tests are shown in Table 20. The Haifa result for F is 5.23 which is higher than critical value of $F(1,364) = 3.86$, at $p = 0.05$, obtained from the tables of F values. This enables us to reject the null hypothesis that there is no difference in the means. However the figure obtained in Edmonton (0.72) is too small to make this assumption (critical value $F(1,73)$, $p = 0.05$ is 3.98).

5.3 Correlation of IQ and the Test Scores

The correlations of IQ with the various test scores are shown in Table 21. The calculations are for the pre-test and the post-test scores in the drawing test, the

Table 19

Error Frequency: Total-Score Pre-test

Quartiles	Haifa Experimental	Haifa Control	Edmonton Experimental	Edmonton Control
$-\infty<e_{ik}\leq -Q$	52	36	13	10
$-Q<e_{ik}\leq 0$	45	41	11	8
$0<e_{ik}\leq Q$	60	49	11	8
$Q<e_{ik}<\infty$	47	36	11	11
Square root σ of the pool variance	18.6	17.8	15.4	15.8
$Q\equiv 0.6745\sigma$	12.6	12.0	10.4	10.7
χ^2 (3)	2.62	2.79	2.61	2.47

Notes: 1. The quartiles are $\pm 0.6745\sigma$, where σ is the square root of the pool variance.
2. The critical value of $\chi^2_{0.05}$ (3) is 7.81.

Table 20

Analysis of Variance: Total-Score Post-test

Source	df	SS	MS	F-Ratio
Haifa: Experimental-Control				
Between	1	208.61	208.61	
Within	364	14519.93	39.89	
				5.23*
Edmonton: Experimental-Control				
Between	1	21.87	21.87	
Within	73	2213.82	30.32	
				0.72**

*$F(1,364)$, $p = 0.05$, is 3.87

**$F(1, 73)$, $p = 0.05$, is 3.98

Table 21

Correlation of Test Scores with IQ

Tests / IQ with: \ Group	Haifa Experimental	Haifa Control	Edmonton Experimental	Edmonton Control
Drawing - Pre-test	0.43	0.37	0.33	0.54
Drawing - Post-test	0.40	0.40	0.31	0.45
Multiple-Choice Pre-test	0.47	0.37	0.42	0.51
Multiple-Choice Post-test	0.44	0.20	0.29	0.55
Total Score Pre-test	0.45	0.37	0.44	0.62
Total Score Post-test	0.46	0.48	0.32	0.55

multiple-choice test and for the total score. The correlations are all positive and of the same order of magnitude.

For the Haifa experimental group little variation was found in the correlations between IQ and the results of the tests. The correlation coefficients vary between 0.40 and 0.47. In the pre-tests, the Haifa control group had a constant correlation, 0.37 for all three test situations. However, in the post-test, the correlation coefficient varied between 0.20 for the multiple-choice test to 0.48 for the total score.

In Edmonton the figures for the experimental group were on the whole lower than those in the control group. The correlations for the various tests in the experimental group vary from 0.29 to 0.44, whilst in the control group the range is from 0.45 to 0.62.

The variations are however small. Thus no significant effect was found in the correlation of score to IQ.

5.4 Level of Geometrical Maturity

"Geometrical Maturity", in the work of Boe (1966) and Bober (1973) means the attainment of a perfect score in the sectioning tests. Boe found that seven of her 72 subjects in grades 8 to 12 (9.7%) attained a perfect score on the drawing tests and three (4.2%) on the multiple-choice tests. None of Bober's 422 students (from grades 7, 8 and 9) had reached this level of Euclidean geometrical maturity.

The present study confirms the above observations. In the drawing pre-test in Haifa no student attained a perfect score. In the post-test nine of the 204 subjects (4.4%) in the experimental group and two of the 162 subjects in the control group (1.2%) attained the maximum score of 16 points. In the multiple-choice test for the experimental group, two students had a perfect score in the pre-test (1.0%), while four (2.0%) attained this score in the post-test. No student in the control group attained full marks in either test. Two subjects attained a full 32 points for the two tests together. They were both for the post-test in the experimental group.

No Edmonton student attained a perfect score in either of the pre-tests or in the multiple-choice post-tests. In the drawing post-test, two students from the experimental group (4.9%) and one from the control group (2.9%) attained a perfect score (Table 22).

Davis claims that, particularly for the children aged 11 to 13, it is not reasonable to demand a full score. He suggests a more reasonable definition of Euclidean maturity would be a total score of 24 (and not of 32) on the tests. Even using this level none of Bober's students had reached the level of geometrical maturity.

In Haifa 43 of the experimental-group students (21.3%) and 36 of the control group (22.2%) attained a total score of 24 or more in the pre-test. In the post-test, 106 (52.5%) of the experimental group and 75 (42.3%) of the

Table 22

Students Attaining High Scores

	Haifa Experimental No.	Haifa Experimental %	Haifa Control No.	Haifa Control %	Edmonton Experimental No.	Edmonton Experimental %	Edmonton Control No.	Edmonton Control %
Scores of 16 in either pre-test*	2	1	0	0	0	0	0	0
Scores of 16 in either post-test*	13	6.4	2	1.2	2	4.9	1	2.9
Score of 32 in the pre-test*	0	0	0	0	0	0	0	0
Score of 32 in the post-test*	2	1	0	0	0	0	0	0
Total Score of ≥ 24 in pre-test**	43	21.3	36	22.2	6	14.6	6	17.6
Total Score of ≥ 24 in post test**	106	52.5	75	42.3	17	41.5	11	32.4

*Geometrical maturity, according to Piaget's definition.

**Geometrical maturity, according to Davis's (1970) definition.

control group attained a score of 24 or more. The corresponding figures for Edmonton were: for the pre-test 6 (14.6%) and 6 (17.6%) and for the post-test 17 (41.5%) and 11 (32.4%).

The z-score was calculated for the mean scores (see Tables 23a,b), using a "defined mean" of 16 (i.e. 100%) for the drawing test and the multiple-choice test *separately*, and a "defined mean" of 32 for the two tests together (Column 3, Tables 23a,b). The z-score is recalculated using Davis's criterion — a defined mean of 12 or 24, i.e. 75%. (Column 4, Tables 23a,b).

The deviations from the defined mean are so large that the probability of attaining the Piagetian maturity is nowhere significant. However, the probability of attainment of Davis-level maturity (75%) is significant for some tests and for some groups — specifically for the Haifa experimental group in the drawing post-test (at $p = 0.05\%$) and in the total-score post-test (at $p = 0.01\%$), and for Edmonton in the drawing post-test for the experimental group and for the control group (both at $p = 0.05\%$).

The marginality of this significance test is, at first glance, surprising. In the Haifa pre-test over 20% of the students were Davis-mature and in the post-test over 40%. In Edmonton the proportions were only slightly lower. The explanation seems to lie in the fact that the remaining students attained much lower scores.

Table 23a

Determination of z-scores: Geometrical Maturity

	Mean	S.D.	z-score[1] "defined" mean 100%	z-score[2] "defined" mean 75%	Probability (p)
Haifa: Experimental, N = 204					
Pre-test					
Drawing	55.2	21.2	30.1	13.3	<0.004
Multiple-Choice	61.7	18.6	29.4	10.2	<0.004
Total Score	58.5	18.6	31.8	12.7	<0.004
Post-test					
Drawing	73.8	18.2	20.6	0.94	0.27*
Multiple-Choice	69.7	19.7	22.0	3.8	<0.004
Total Score	71.8	18.1	22.3	2.5	0.02**
Haifa: Control, N = 162					
Pre-test					
Drawing	55.1	20.5	27.9	12.4	<0.004
Multiple-Choice	59.8	18.2	28.1	10.6	<0.004
Total Score	57.6	17.8	30.3	12.4	<0.004
Post-test					
Drawing	69.2	21.7	18.1	3.4	0.009
Multiple-Choice	54.5	23.8	19.0	5.6	<0.004
Total Score	67.0	21.5	19.5	4.7	<0.004

* Significant at 0.05

** Significant at 0.01

(1) Total: 16 for Drawing and Multiple-choice, 32 for Total Score.

(2) Total: 12 for Drawing and Multiple-choice, 24 for Total Score.

Table 23b

Determination of z-scores: Geometrical Maturity

	Mean	S.D.	z-score [1] "defined" mean 100%	z-score [2] "defined" mean 75%	Probability (p)
Edmonton: Experimental, N = 41					
Pre-test					
Drawing	53.2	21.5	13.9	6.5	<0.004
Multiple-Choice	63.7	13.7	17.0	5.3	<0.004
Total Score	58.2	15.4	17.4	7.0	<0.004
Post-test					
Drawing	72.3	19.1	9.3	0.91	0.27*
Multiple-Choice	63.6	15.9	14.7	4.6	<0.004
Total Score	67.9	16.4	12.5	2.8	0.008
Edmonton: Control, N = 34					
Pre-test					
Drawing	53.9	20.5	13.1	6.0	<0.004
Multiple-Choice	62.5	16.8	13.0	4.3	<0.004
Total Score	58.2	15.8	15.4	6.2	<0.004
Post-test					
Drawing	70.2	20.0	8.7	1.4	0.15*
Multiple-Choice	58.8	18.8	12.8	5.0	< 0.004
Total Score	64.5	17.6	11.8	3.5	0.009

* Significant at 0.05

** Significant at 0.01

(1) Total: 16 for Drawing and Multiple-choice, 32 for Total Score.

(2) Total: 12 for Drawing and Multiple-choice, 24 for Total Score.

5.5 Results of the Study

In chapter IV the following question was posed: Does the study of the unit "Making Rectangular Solids" significantly affect the performance of sixth- and seventh-grade children in a geometrical-maturity test?

Two subsidiary questions were also investigated:

(1) Did studying the unit have the same effect on the score in the geometrical-maturity test in Haifa, in Israel as in Edmonton, in Alberta?

(2) Was there any correlation between the IQ of the subject and his scores on the geometrical-maturity test?

There was no significant difference between the performance of the experimental and control groups in the pre-tests. Nor was there any significant difference between the Haifa and Edmonton samples. There was no significant deviation from normality.

In the post-tests most of the mean scores increased. The exceptions were the multiple-choice tests for the two Edmonton groups. For the experimental group the fall was negligible, from 63.7% to 63.6%. The control group fell from 62.5% in the pre-test to 58.8% in the post-test (see Table 11, p. 94). Overall there was an improvement in the performance of the students of both the experimental groups (the students who studied the material), compared with the corresponding control groups. However, while this improvement was significant for the sixth-grade Haifa students, it

was not significant for the seventh-grade Edmonton sample.

There are several possible reasons why the experiment showed a greater gain in Haifa than in Edmonton. Two obvious ones are:

(a) The type of material was novel for the Israeli children. They are accustomed to traditional frontal teaching and were taken out of their usual routine.

(b) The teachers took a great deal of interest in the progress of the experiment. They regarded it as a privilege to be invited to participate.

Since all the changes in scores were marginal, these and other spurious effects may have biased the results. The conclusion that the unit improves the performance on the geometrical-maturity tests can only be tentative.

The correlations found between the IQ and score in the geometrical-maturity test ranged between 0.2 and 0.6. The experimental groups performed more consistently than did the control groups. In Haifa the range for the experimental group was from 0.40 to 0.47; in Edmonton it was from 0.37 to 0.48 (except for the multiple-choice post-test, see Table 21, p. 108).

There was no consistent change in the correlations before and after studying the unit. One might have expected that the brighter children would have gained more from the unit. However the structure of the sectioning tests is

biased against detecting any such effect. Students who attain a high pre-test score have little room for improvement.

Chapter VI

Conclusions and Implications

6.1 Summary of the Study

The study was based on a teaching unit "Making Rectangular Solids" (Kuper and Walter, 1978). The unit was designed to give the child concrete experiences in solid geometry by building a set of rectangular prisms, examining them, and playing with them. The intention of the unit was to provide a series of exercises to enable the student to discover the properties of the solids for himself, to examine some special lines and planes in space, and to serve as an introduction to the concept of volume.

The study was conducted in Haifa in Israel and in Edmonton in Canada. In Haifa 204 students formed the experimental group, and 162 the control group. In Edmonton the corresponding numbers were 41 and 34, respectively. A pre-test was administered to all the students at the start of the experiment. The experimental groups then studied the unit. At the conclusion of the work, the groups were all re-tested with a post-test, similar in structure to the pre-test. An achievement test was also administered.

The method of testing was to ask the student to identify sections of geometrical solids. The students first drew the sections freehand, and then tried to choose the correct drawing from a set of five alternative drawings.

In the pre-test the solids used were: a cube, a triangular prism, a non-rectangular parallelepiped, and a cone. In the post-test the solids were: a rectangular prism, a four-pointed star-shaped prism, a circular cylinder and a square-based pyramid. There were four sections for each of the solids - making in all 16 sections in each of the tests. These sections were labelled: longitudinal A, longitudinal B, transverse, and oblique.

The unit was taught by the regular class teacher. In Haifa the teachers were supervised by the author of this report. In Edmonton the teachers received a short explanation, a full teachers' guide and notes on the lessons.

The main purpose of the study was an attempt to answer the question: Does studying the unit "Making Rectangular Solids" significantly affect the performance of sixth- and seventh-grade students on the geometric-maturity test? Two subsidiary questions were also examined. Firstly the effect of the unit on two different types of population was investigated - the sixth-grade students in Haifa and the seventh-grade students in Edmonton. Secondly the correlation coefficient of IQ with the score on the tests was calculated.

6.2 Discussion of the Results

The main purpose of the study was to see if the unit "Making Rectangular Solids" could affect a student's performance in the geometrical-maturity tests. However before

discussing the conclusions and implications of the study, some comments must first be made. These comments will apply mainly to the Haifa experiment where the author had direct contact with the progress of the study.

(1) The unit was designed to give the child particular geometrical experiences of discovery, manipulation and building the models. It was hoped that the child's level of geometrical maturity, as measured by the tests, would be increased. However, eight to ten lessons may not be sufficient to make an impact on the child - although most children greatly enjoyed the experience.

It is difficult to see how the unit can be fully effective if it is studied too quickly. The child will not have adequate time and opportunity to assimilate the experiences. For maximum effectiveness the lessons should probably be spaced over a longer period of time.

(2) The spirit of the writing of the unit was one of "discovery learning", but most of the teachers in Haifa were not used to this teaching style. It is the author's opinion that they made an honest attempt to teach in this way (at least as stated in their written reports and when the author was present in the classroom), but a lifetime of teaching habits is hard to change overnight. It cannot be claimed that the unit was taught in *all* the schools *all* the time in the way it was designed.

(3) The classroom work was done by the children in groups of three or four. Thus there is no certainty

that *every* child benefited from *all* the experiences. It was hoped at the beginning of the experiment that the sample would be sufficiently large to allow for statistical fluctuations due to this particular defect. However, in spite of the fact that over 500 children participated in the Haifa experiment, only 366 were included in the final sample and 204 of these studied the unit. The experiment would have been more meaningful if it had been possible to be sure that all the students had taken an active part in all, or at least most, of the activities.

(4) The tests were conducted for the class as a whole and not with each child individually. Both the teacher and the tester were present. Nevertheless one cannot have as much confidence in data obtained in this way as in data collected in an individualized examination, as in the experiment of Boe (1966). Even testing in small groups, as Davis (1969) did, is greatly to be preferred over whole-class testing.

For example, in the classroom situation a child might ask for the tester to repeat a particular cut, or he might ask to see the solid from a different aspect. It is clear that, if one child is in some doubt about his response, others might be influenced by his hesitation. A second example of influence from the other class members could occur when an "easy" section (e.g. the sections of the cube or the rectangular box, or the transverse sections of the triangular prism) followed a "difficult" section (e.g. the

oblique section of the cone, cylinder, or pyramid). It was sometimes possible to hear the sigh of relief - AH! - or even an enthusiastic comment. This is unavoidable in the classroom situation, especially when the students are keenly interested in what they are doing.

When 35 to 40 children are sitting in pairs in the classroom (the normal seating arrangement in Israeli schools) other spurious influences can occur. The experimenter was not aware of any conscious copying, and every effort was made to prevent it. But she cannot be certain that some unintentional copying did not occur, particularly in the multiple-choice tests. This is an inherent danger when testing the entire class as a unit rather than testing the students individually.

(5) The structure of the tests made it difficult to measure the improvement in performance of high-ability students. If a student obtained a score of 14 to 16 out of a total of 16 points in the pre-test, he could not gain more than one or two points on the post-test. He had nowhere to go but down. However an average or a weak student who had a pre-test score of under 8, could gain up to 8 points. This means that the analysis of the gain score is biased against the student of high ability, who had performed well in the pre-test.

(6) In Haifa the scores had been cross-checked so that a particular drawing response was compared with its multiple-choice response, to see if the drawing had been

"read" correctly. However as only the score sheets were obtained from Edmonton this check could not be made. In addition there is always the possibility that two different people could record the drawing scores differently. In fact the rules laid down by Drost (1977) seem to be unduly rigid for tests carried out in the classroom situation, when compared with tests performed by individual interview. Students, influenced by their peers, may feel that they are under pressure of time, and in their haste they may draw less accurately. In the author's opinion, one can have full confidence only in results obtained by individual interview.

(7) The experimenter's results do not confirm the observations of Drost (1977, p. 150) that there is an increase in test means in the order:

(1) pre-test drawing,

(2) pre-test multiple-choice,

(3) post-test drawing,

(4) post-test multiple-choice, i.e., the increase paralleled the order of administration of the tests.

In the present study the following order was found for both experimental and control groups in Haifa (the order of presentation being the same as in Drost's study):

(1) pre-test drawing,

(2) pre-test multiple-choice,

(3) post-test multiple-choice,

(4) post-test drawing.

For the Edmonton sample the order of the increase of test means for both groups was:

(1) pre-test drawing,

(2) post-test multiple-choice,

(3) pre-test multiple-choice

(4) post-test drawing.

(8) The author's Haifa data confirmed Drost's observation that even in the control group the post-test scores were higher than the pre-test. A possible explanation is that the pre-test in effect acted as training for the post-test. This point is dismissed by Drost on the grounds that there was a time lag of 4 to 7 weeks (in Haifa this was sometimes as long as three months). He believed that this lapse of time was too long for the training effect to be important. He also claims that the results of the first test were not returned to the students, so that they did not know whether or not they had been successful in it.

The author of this report does not agree with Drost's observations. It was apparent that some learning had taken place; it was observed that the time required to administer the post-test was considerably shorter. There was no need to explain in detail what was expected of these students. As soon as the experimenter appeared with the box of solids, the children were ready to start. Although the students did not know the results of the pre-test, they had not been isolated and had discussed the tests with each other and

with their class teacher after the pre-test session.

(9) Bober (1973) claimed that the two tests were equivalent in difficulty. The author agrees with Drost's repudiation of this statement. In the pre-test there were four "difficult" sections: the oblique sections of the cube, triangular prism and cone, and the longitudinal B section of the cube. In the post-test, only the oblique sections of the rectangular prism and the circular cylinder were difficult. The sections of the star-shaped prism were difficult to draw accurately, but were recognizable as the intended shape. Thus, overall, the pre-test seems to be harder than the post-test, which might account for the improvement in the score of the control group.

In the light of these observations the following conclusions are drawn:

(1) There was no significant difference between the mean score for the experimental group compared with the control group for the drawing test, the multiple-choice test, and the total score. These scores were also normally distributed. This was true both in Haifa and in Edmonton.

In the Haifa post-test, there was a significant difference in the mean score of the three tests (drawing, multiple-choice and total score) of the experimental group over the control group. Thus the unit had an effect on the performance of these students.

(2) Although the Haifa students were nearly a year younger, they performed better than did the Edmonton stu-

dents. This difference may have been due to external factors, such as the personal interest of the experimenter in seeing that the teachers were adequately prepared, informed and encouraged. It may have been due in part to the interest shown in the unit by the students, as the teaching methods used and the general approach were novel and exciting for the Haifa students. It will be interesting to learn if a study, to be performed in an Arab village in Galilee, shows similar effects.

(3) No real evidence was found that the ability (measured by the IQ) had an effect on achievement in the unit. The correlations of score with IQ were all low. However the nature of the tests had a built-in bias against finding such an effect. The student who gained a high score in the pre-test was not able to make a significant percentage gain.

6.3 Implications and Recommendations

6.3.1 Implications of the Study

This study was motivated by a desire to improve teaching of geometry and to overcome the prevalent anti-geometric prejudices of many teachers. This prejudice is not shared, at first, by the children. Many children can get excited about shapes and solids and enjoy model building. They also derive much artistic and aesthetic pleasure from some aspects of physical geometry.

It has already been stressed that the teaching of measurement of areas and volumes is a problematic topic in the curriculum. If the first approach to the study of these concepts is through activities of the type described in this study, it is more likely that the student will be able to distinguish between them.

However it is clear from the results of this and similar studies that the concepts of area and volume are taught to children at a stage when not all of them have reached full geometrical maturity. As a consequence two possible courses are open. Firstly, educators should ensure that sufficient manipulative experiences be available to the student before and during the study of area and volume. Secondly, it might be more efficient to delay the teaching of these concepts until grade 8 or 9, when the child is more mature.

The correlation of IQ with the score in the tests was not found to be significant. However the tests are, in the author's opinion, not sufficiently sensitive to be able to make any conclusions on this point.

On the whole, the Israeli children performed better than did the Edmonton children, although they were a full year younger. That the increase was in part due to the novelty of the situation cannot be discounted. It will be instructive to learn if this trend is maintained when the whole Israel geometry curriculum is reformed and the sixth-grade students are tested after many such experiences.

6.3.2 Recommendations for Improved Sectioning-tests

1. From the results of Drost (1977), it is clear that the geometrical-sectioning tests are still not adequate. According to Drost, the pre-test is more difficult than the post-test. While this point has not been specifically investigated here, the author's general impression supports Drost's contention. This statement could be verified by subjecting the two groups of subjects to the tests, but reversing the role of the pre- and post-test for one of the groups. If this imbalance should prove significant, the tests should be modified.

2. Some of the sections proved too easy; in other words, practically all of the students gave a correct response. These over-easy questions include the square sections of the cube and the rectangular sections of the rectangular prism. While having easy tasks is not in itself damaging, it means that an opportunity of asking more searching questions has been lost. One might remove some of the easier sections from the tests and in their place use the cuts of the regular octahedron. The simpler sections of the cube could then replace the sections of the octahedron in the demonstration session, before the pre-test.

3. The distractors used in the multiple-choice test were those described by Boe (1966). She derived them from the incorrect drawings by her subjects. However some of these subjects were only four to six years old, and made

errors which 11-year-old children are unlikely to make. The choice of distractors should be carefully reconsidered, and a more plausible set of distractors prepared.

4. Drost claims that there is little difference between the results of classroom and individual testing. He therefore suggests that the tests could be used as a measuring instrument for classroom use. For the reasons presented in section 6.2, the author does not agree with this view. It is admittedly possible to test larger numbers more quickly in the classroom. However the children's drawings are often ambiguous. In individual testing, the tester sees the drawing immediately, and where it is ambiguous, the child can be asked to clarify his intentions.

6.3.3 Recommendations for Further Study

In the light of this study, the author feels that the activities described in the unit are not enough to significantly improve the geometrical maturity of the students. In future this unit will be used as part of the grade-5 curriculum in Israel. Four new units will be used in grade-6. They are:

1. Area and perimeter,
2. The regular (Platonic) solids,
3. The prisms, including the cylinder as a special case,
4. The pyramids, including the cone as a special case.

All the new geometry material produced by the Curriculum Centre in Israel is based on activity by the students.

The central theme in the unit on the Platonic solids ("Let's Make Regular Solids") is to discover that five, and only five, regular solids exist. In the units on prisms and pyramids the volumes of the solids are investigated for sets of solids with equal-area bases but differing heights, and of bodies of equal heights but different base area.

It is hoped that more extended study will be made, along the lines of the present work, when the units described above are completed. It will then be possible to compare the effect of this material on the geometrical maturity of students in a variety of situations. For example, the effect on a group of students of studying the material in the space of two or three months could be compared with the results obtained from a group studying the same material over the course of an academic year.

In this proposed study, students would be tested individually, and not in whole-class groups. Both the present tests and modified ones - using improved distractors - should be used. The selection of models between the pre-test and the post-test should also be optimized.

Finally, it is hoped that the material described in this study will stimulate others to prepare interesting material for the geometry curriculum. This might eventually influence teachers not to neglect geometry.

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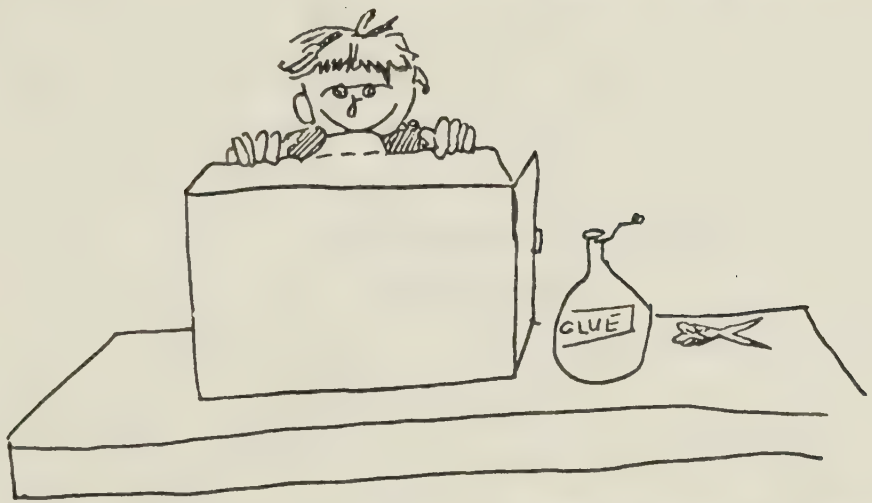
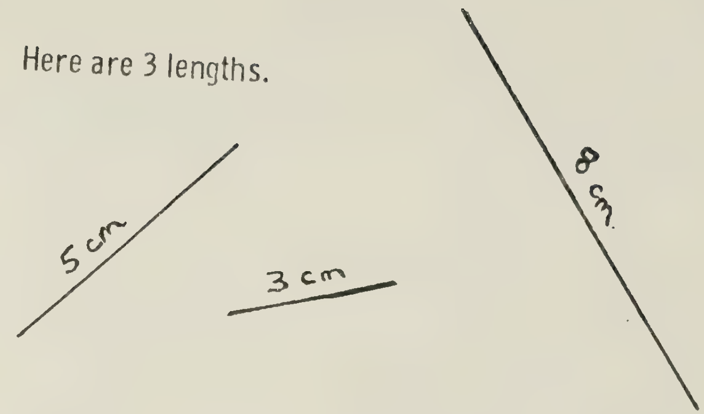
MAKING RECTANGULAR SOLIDS



Marie Kuper and Marion Walter

Here are 3 lengths.

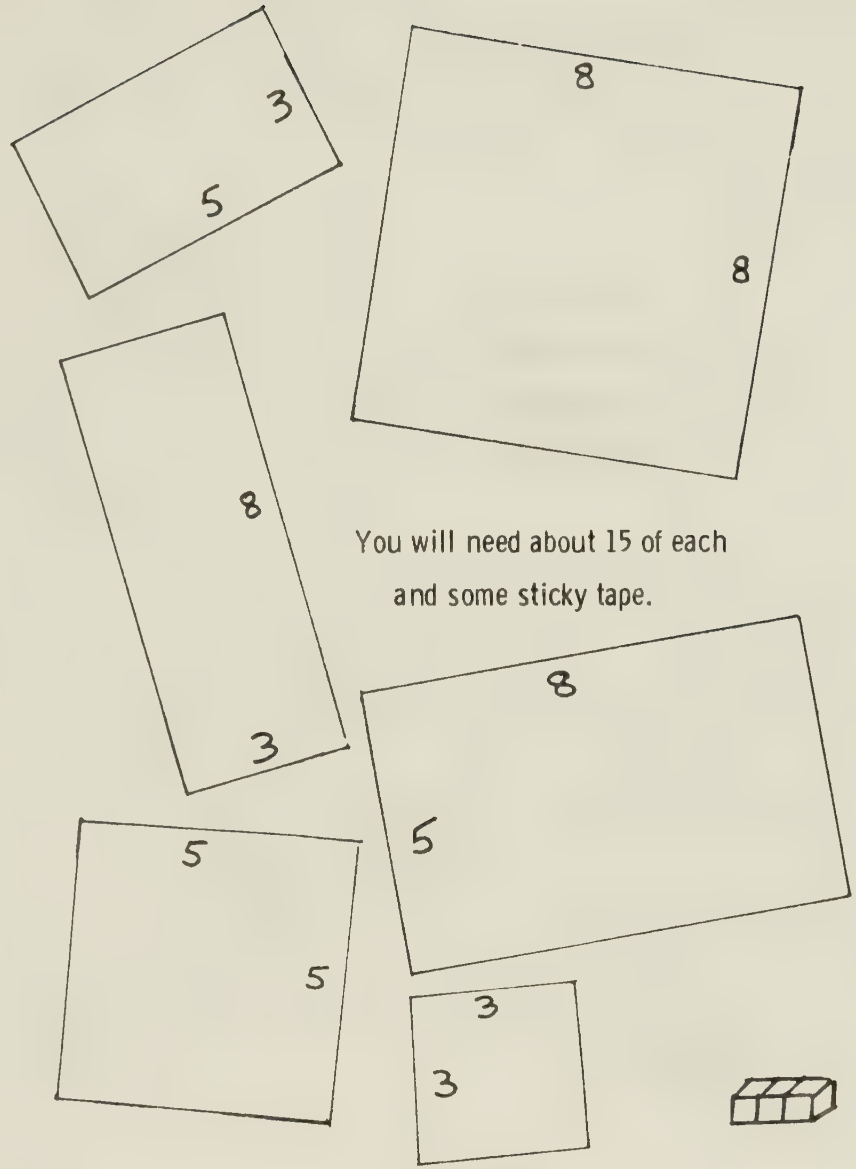


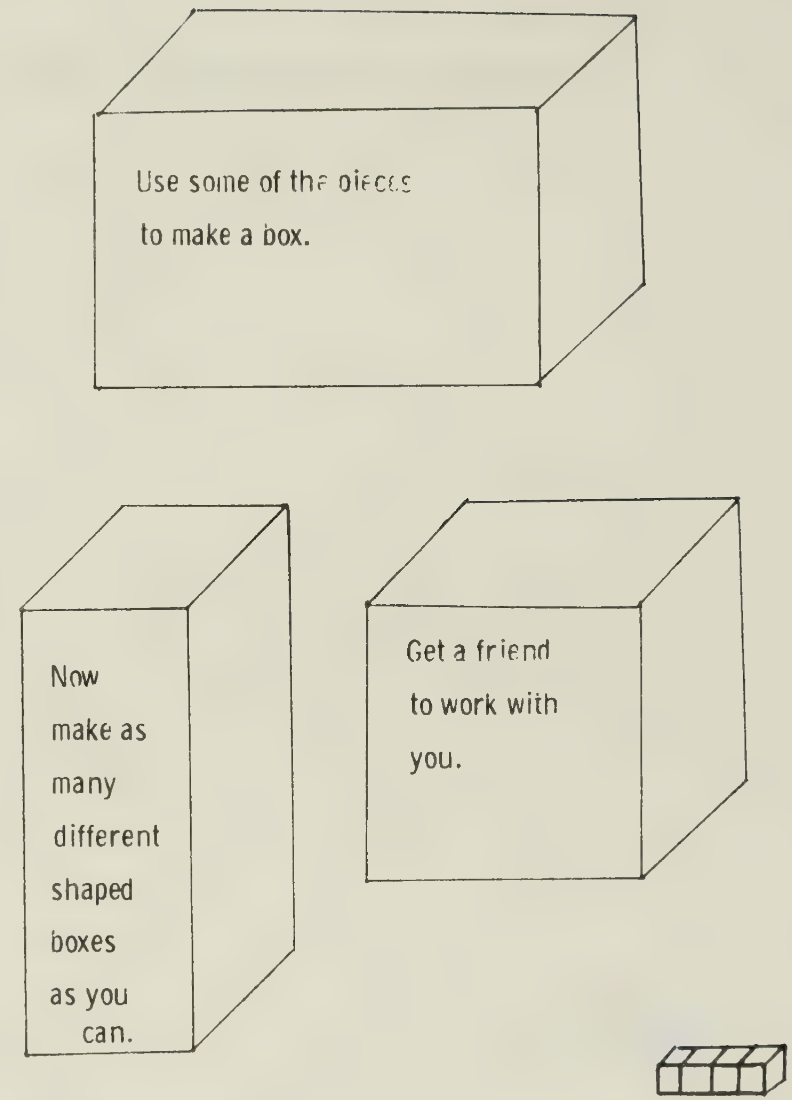
How many different sized rectangles can you make with edges 3 , 5 , or 8 cm?

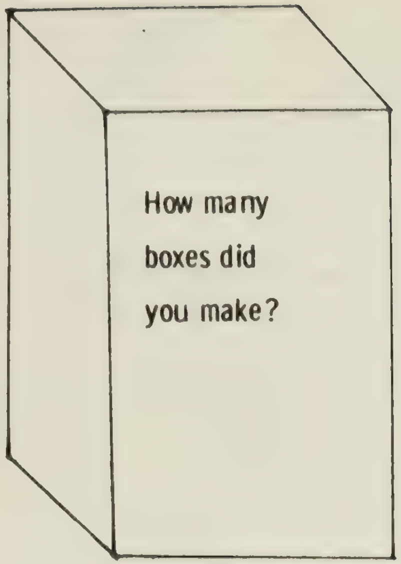
You can use each length as often as you need.

Draw them.

How many did you find ?


3
5
8
8
8
3
You will need about 15 of each
and some sticky tape.
8
5
5
5
3
3

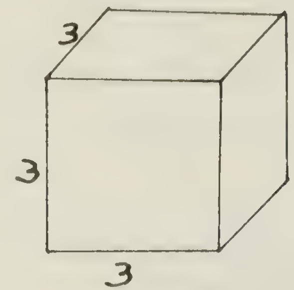

Use some of the pieces
to make a box.
Now
make as
many
different
shaped
boxes
as you
can.
Get a friend
to work with
you.

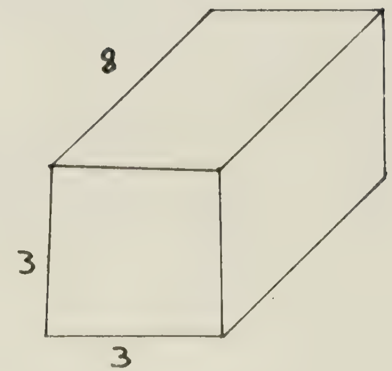


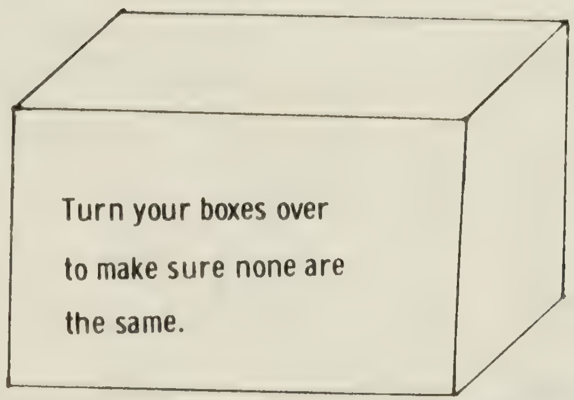
Do you think the pictures on this page show the same box?

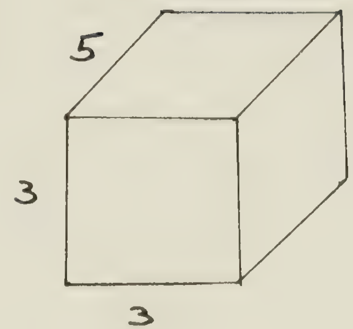
One can make 10 different boxes from the shapes on page 3.

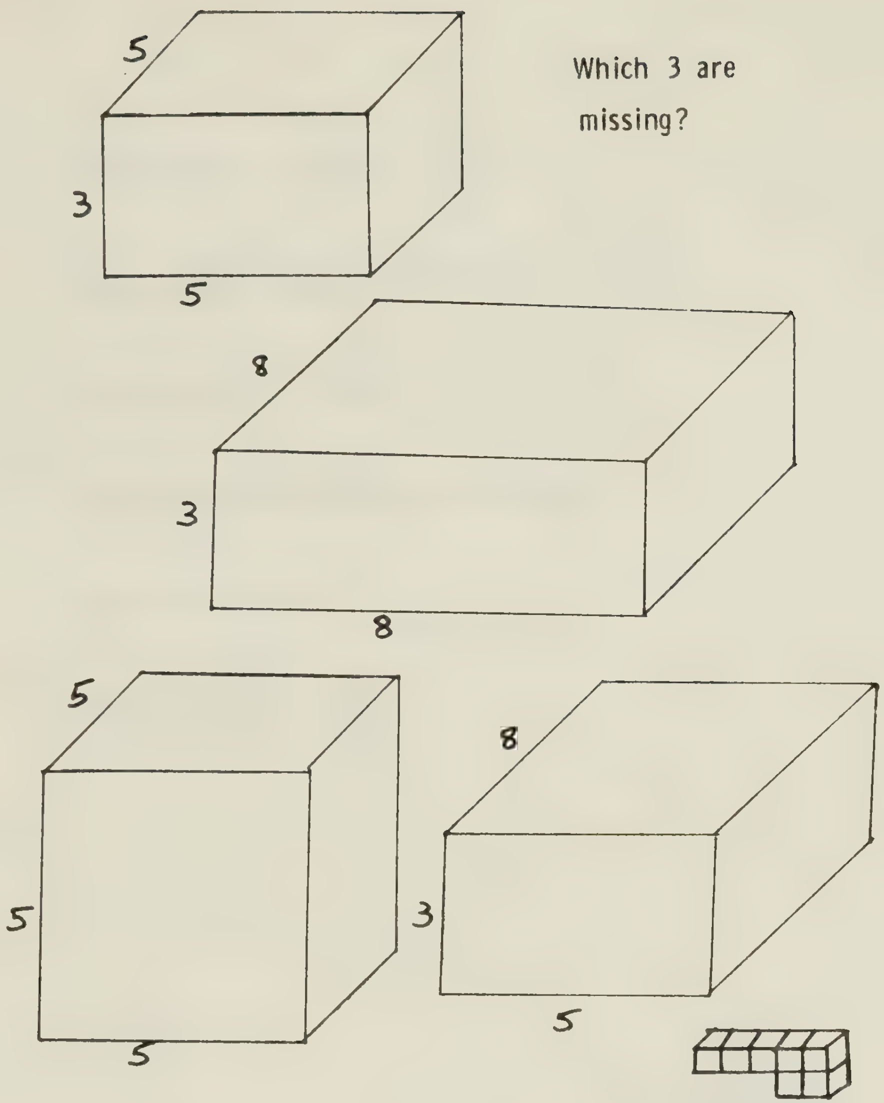
Here are pictures of 7 of them.







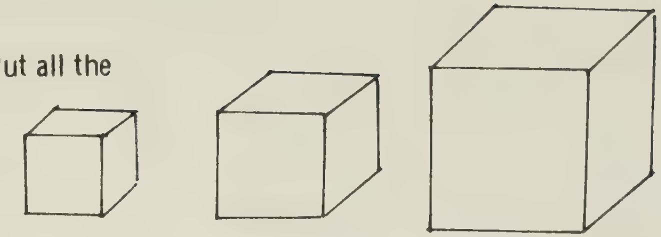




Which 3 are missing?

Take the 10 boxes.

Put all the

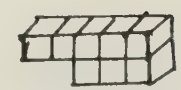


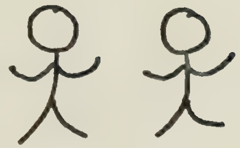
cubes in one set and the non-cubes into another.

Find a different way of sorting the boxes into two or more sets.

Ask someone to describe your sets.

Take turns.



A GAME FOR 

Let your partner look at the 10 boxes on the table.

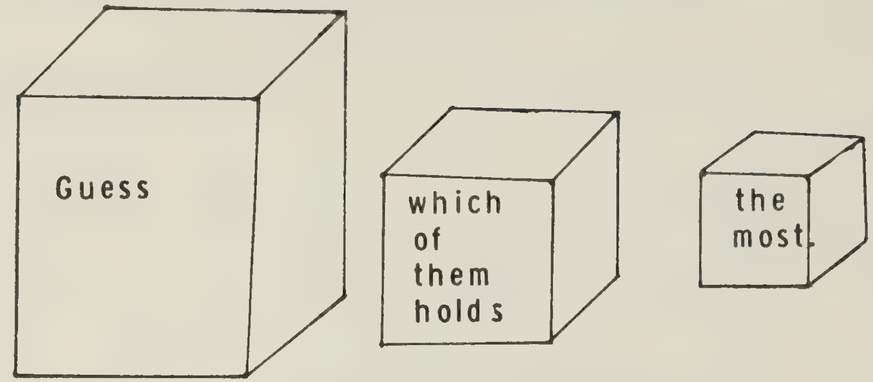
While his eyes are closed hide one of the boxes.

Ask him to try to describe the missing box.

It isn't always easy!

Put it back and take turns.

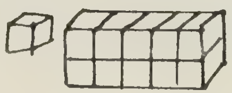
Look at all the 10 boxes.



Arrange the 10 boxes in order from the largest to the smallest.

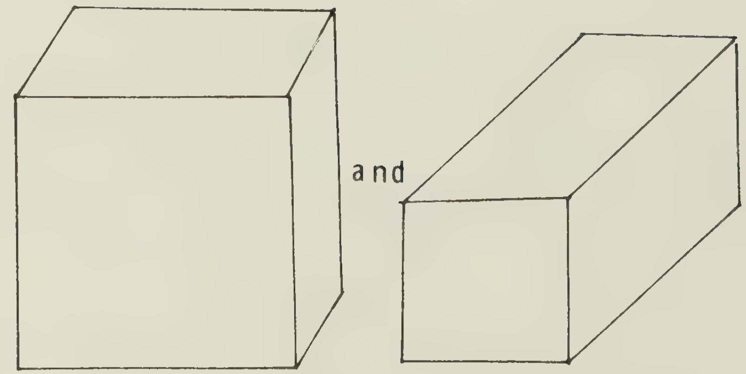
Label each box and record the order you choose.

BOX	SIZE	VOLUME
A	3,3.3	Calculate the volume of each box and complete the table in your notebook.
?	3,5,8	
J	8,8,8	



Compare your table with the order you GUESSED on page 10.

Many people put

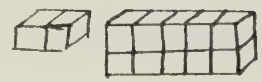


in the wrong order.
Why do you think this is so?

Which is the bigger?

Which boxes did you have in the wrong order?

Are there other "dangerous" boxes?

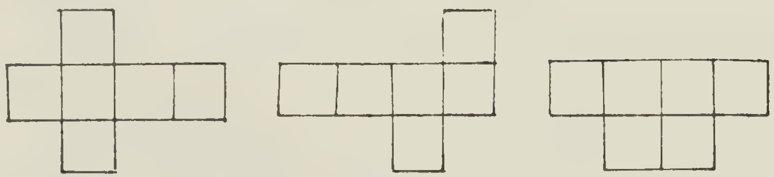


Your boxes were made of

squares and rectangles that were not squares.

How many boxes did you make using only squares?

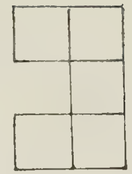
With no squares?



Here are some patterns using only squares.

Will they fold into closed boxes?

Use cut outs from squared paper to check.

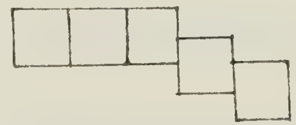


Here is a five-square pattern.

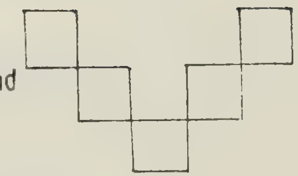
Draw other ways of arranging 5 squares.

is allowed.

but



and

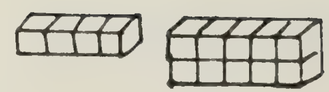


are not.

Use cut out squares to move around if you like.

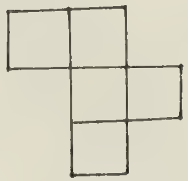
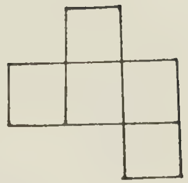
Draw your patterns on squared paper.

How many different patterns did you draw?

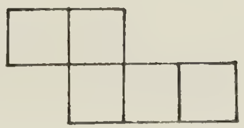
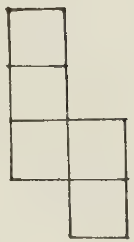


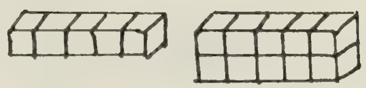
Cut out each of your patterns. Check that they are all different.

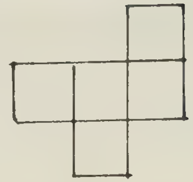
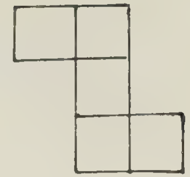
Were you right?

For example,  and 

are the same pattern.

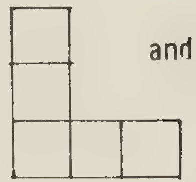
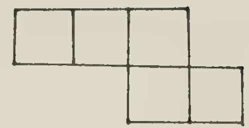
Are  and  the same?

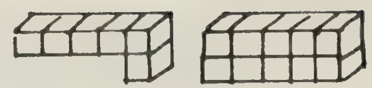


What about  and  ?

Did you find 12 different patterns?

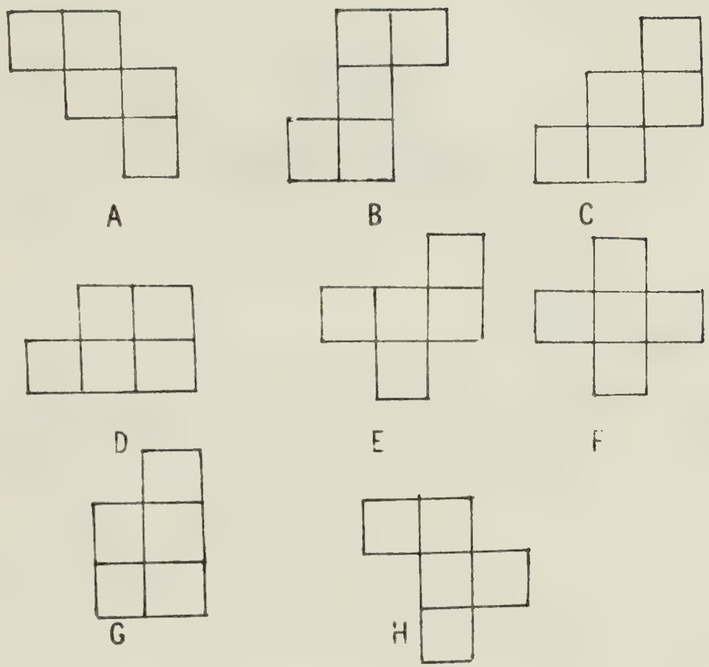
If you did not find all 12, look again.

Many people forget  and 

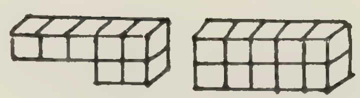


Here are some 5 square patterns.

Which patterns are duplicates?

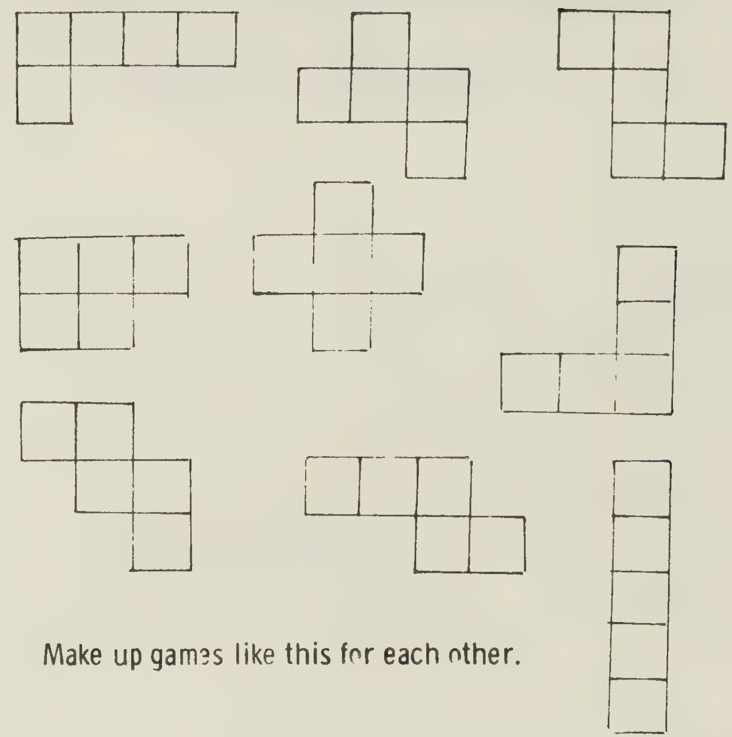


Make up a problem like this for someone else.

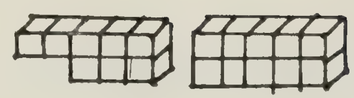


Here are 9 of the 12 patterns.

Draw the 3 missing ones in your notebook.

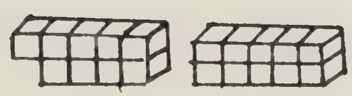


Make up games like this for each other.



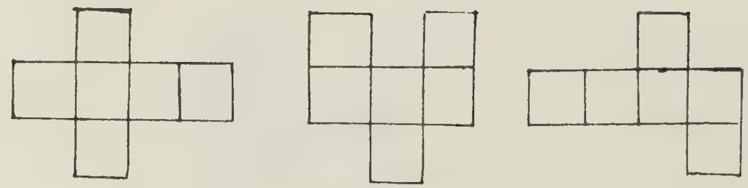
Which of the 12 patterns fold into an <u>open</u> box?

Check your answers by folding.



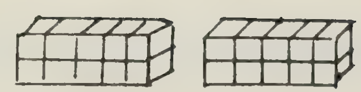
Here are two patterns of SIX squares that fold into boxes with a top and one pattern that doesn't.

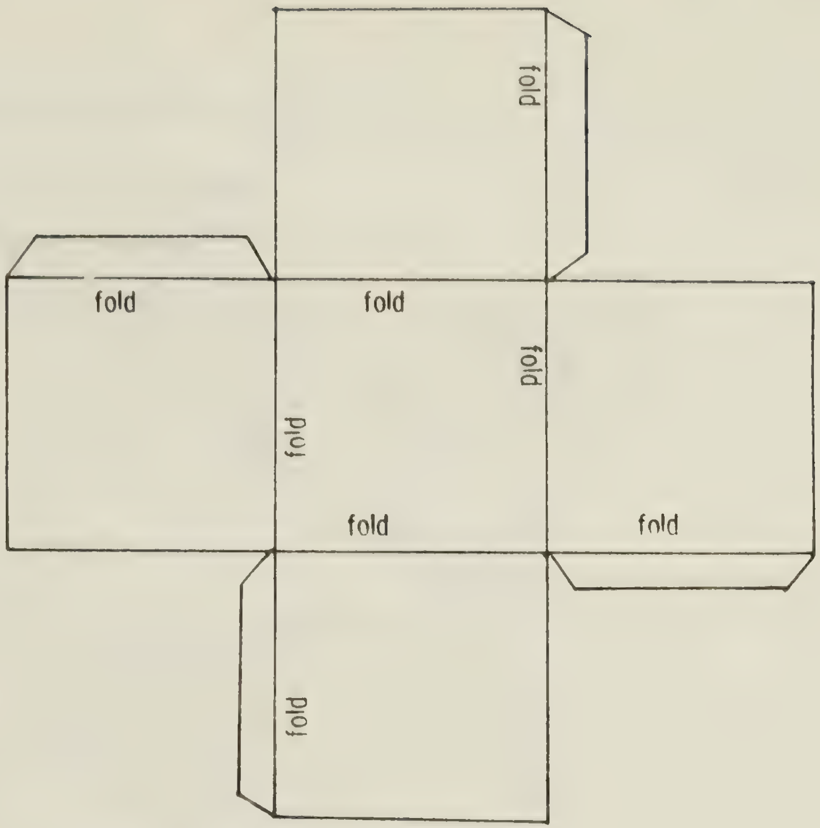
Which ones fold into closed boxes?



Draw other six-square patterns.

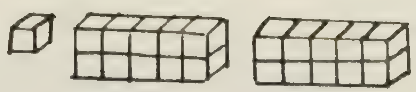
Which of them fold into a box?





Here is a net for building an open cube.

Trace it onto thin card.



Mark along the fold lines with a sharp instrument so that it will fold easily.

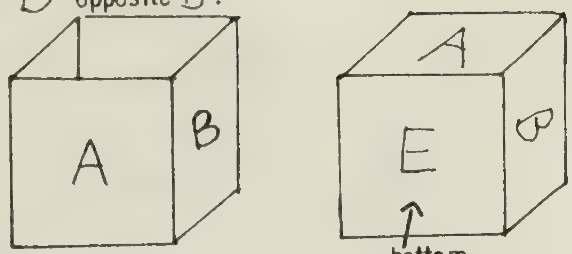
Build the cube by glueing the tags.

How many tags does one need?

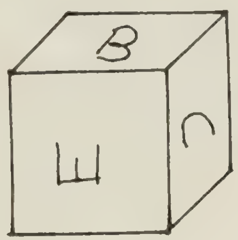
Could they be put along other edges?

Label your open box on the outside.

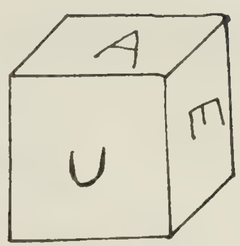
write E on the bottom.
Write C opposite A.
Write D opposite B.

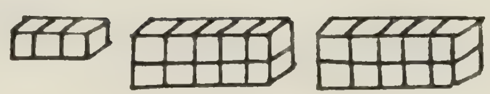


If you turned the box, could it look like this



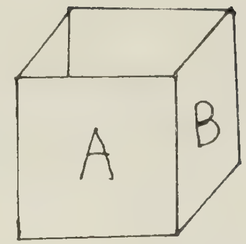
or this?



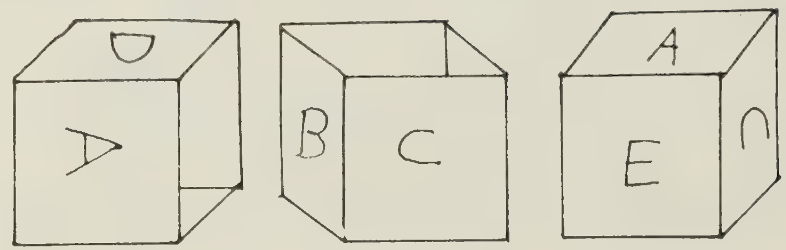


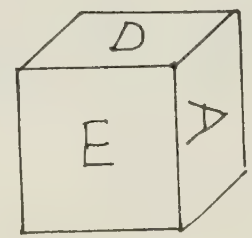
Look at your box.

Can you turn the box to look like the pictures below?



Try to decide just by thinking!





Check by turning the cube.

Instead of using different letters we can use colours.

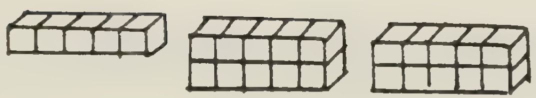
Choose 5 colours.

Colour each face with a different colour.

In how many different ways can you colour your

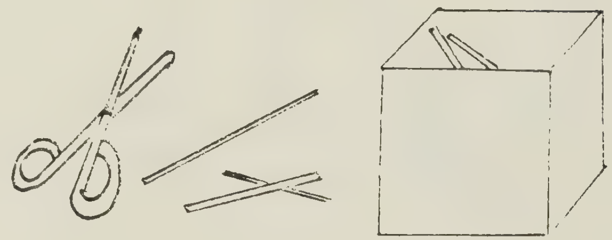
box with 5 colours?

Don't use the same colour twice!



Look at your open box.

Cut straws into pieces 4 cm, 5 cm, 6 cm, 7 cm, 8 cm, 9 cm and 10 cm in length.



Which is the <u>longest</u> piece that will fit into your box (without bending the straw)?

GUESS FIRST ..... THEN CHECK!

These pieces of straw were cut in whole centimetre lengths,

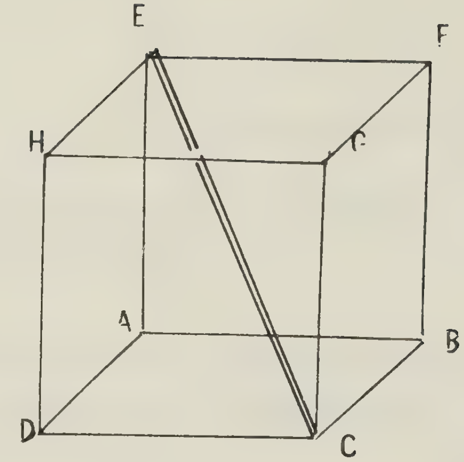
Look at the longest piece of straw that will fit into your box.

What is the biggest piece that you can cut and that still fits into the box?

It need not be a whole number of centimetres.

Try it!

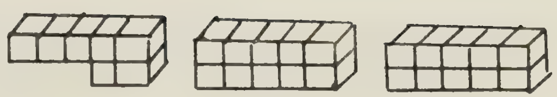
Can you describe where in the box it lies?



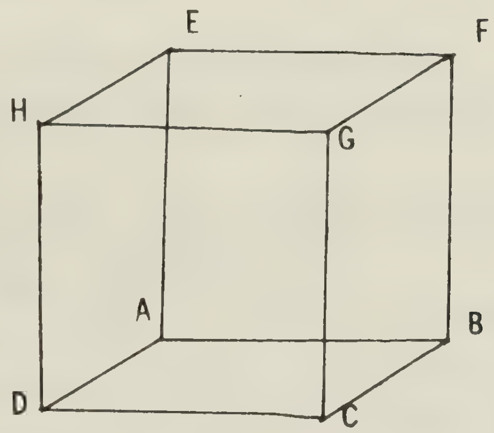
Here is one way in which the biggest stick lies in the box. It goes from corner E to corner C.

Write down how else it can fit into the box.

Try to draw pictures of your results.



Cut the longest straw which will lie flat in the bottom of the box.

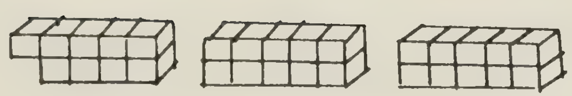


How long is it?

How many ways will it fit?

Draw a picture of you result.

Make up other problems.



A PROBLEM FOR A GROUP OF YOU.

Draw nets for open cubes, like the one on page 21.

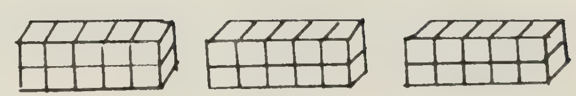
Each of you choose a number and use it for the length of a side (in centimetres).

If you haven't learnt how to construct a square using a ruler and compass, look at the next two pages.

With the new cubes, repeat page 27.

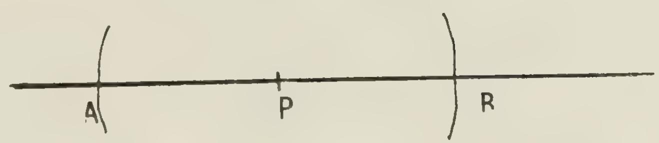
Compare your results.

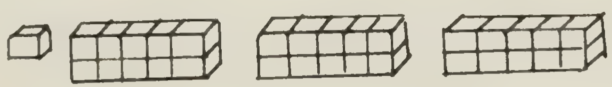
Discuss them with your teacher.

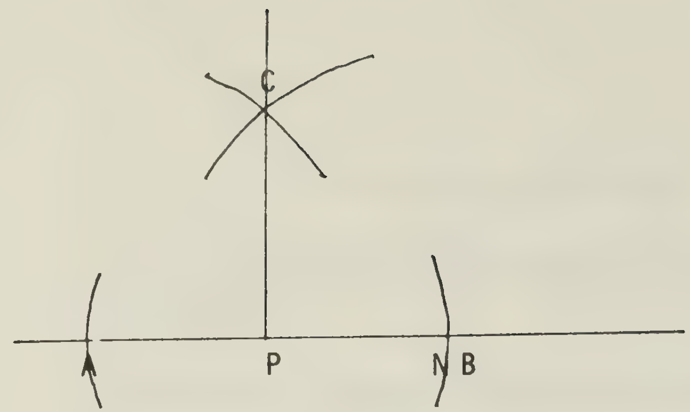


HOW TO DRAW A RIGHT ANGLE WITH ONLY A RULER
AND A PAIR OF COMPASSES.

1. Using a sharp pencil, draw a straight line.
   Don't press hard, as you may want to erase part of it later.
2. Choose a point on the line where you want to put the right angle. Call this point P.
3. Open the compasses 3 or 4 cm. It doesn't matter exactly how much, but choose a comfortable size. (Don't make the opening too small or too big, as then you will find it hard to draw accurately.)
4. Put the point of the compasses on P and draw two arcs cutting the line. Call the left intersection A and the right one B.







5. Now open the compasses WIDER.
   Take A as a new centre and draw a faint circle.
6. Now take B as centre and draw an arc, cutting the circle (whose centre is at A) at C.
7. Join PC. The angle CPB is a right angle.
   (If you like you can now erase any unnecessary lines.)

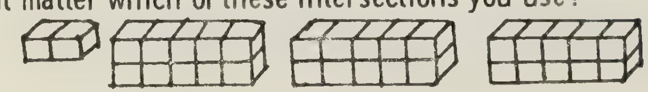
Why must the radius used in step 5 be larger than the radius used is step 4?

Did you need to draw a complete circle?

Does it matter if the circle is drawn about A or about B?

How many intersections C can you find?

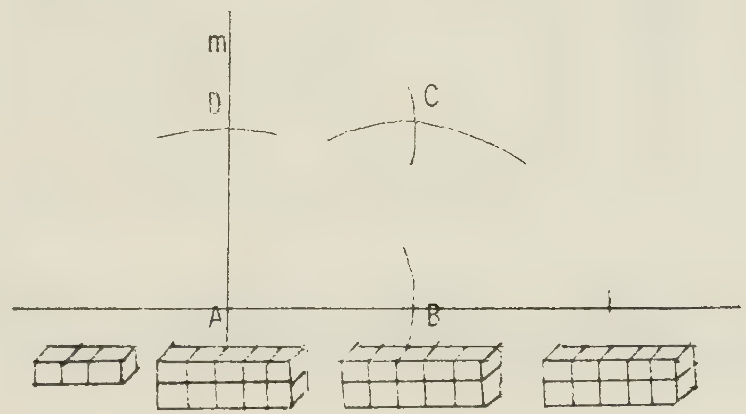
Does it matter which of these intersections you use?



HOW TO DRAW A SQUARE WITH ONLY A RULER AND A PAIR OF COMPASSES.

1. Draw a pair of lines 'l' and 'm' intersecting at right angles. Call the point of intersection A.
2. Choose the length of the side of your square. Set the compasses to exactly this length.
3. Put the point of the compasses on A and mark two arcs, one cutting l at B, and one cutting m at D.
4. Now use B as centre and draw an arc. With D as centre draw another arc, to cut the one you have just drawn at C.

   Be careful not to alter the setting of your compasses!

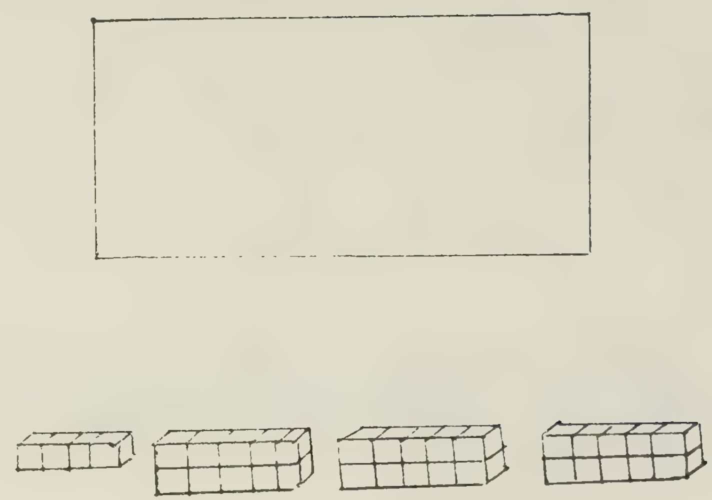


5. Join BC and CD. ABCD is your square.

Is this the only square you could draw with these instructions?

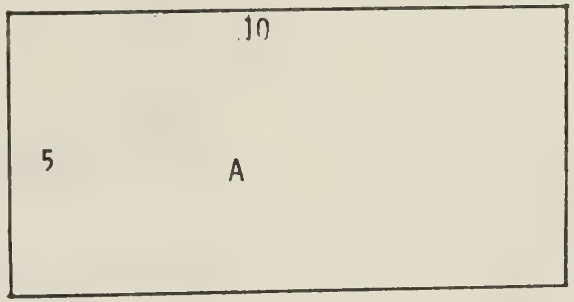
Suppose the arcs you drew in step 4 didn't intersect, what could you do about it?

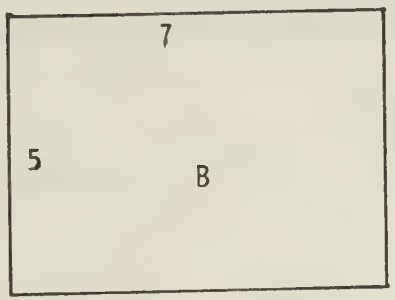
How could you draw a rectangle?

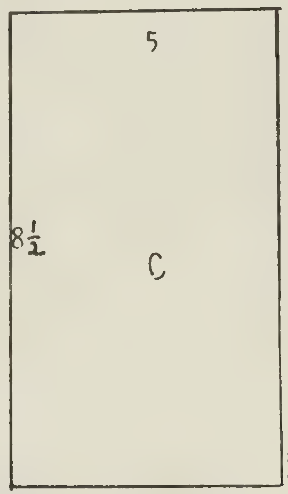


Clearly

will fit into the open box you made on page 21.
which of these rectangles would fit into the box exactly?

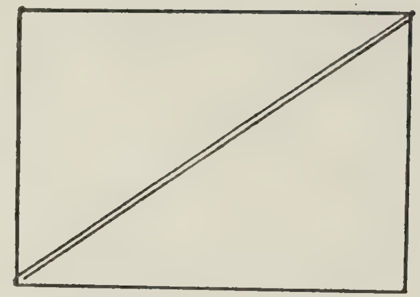






In how many ways does the piece B fit into the box?

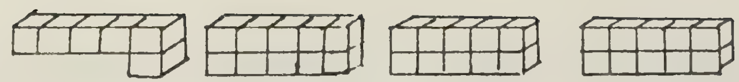
Stick a straw along the diagonal



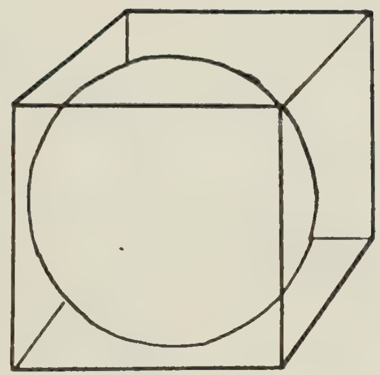
and put it in the box.

How does the length of this stick compare with the longest stick you cut on page 27?

Notice where the stick lies in the box.



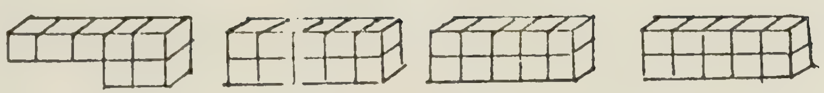
THE OPEN CUBE.



Make a ball from a lump of plasticine.

What is the biggest ball which will fit into your cube without protruding?

How could you find its volume?



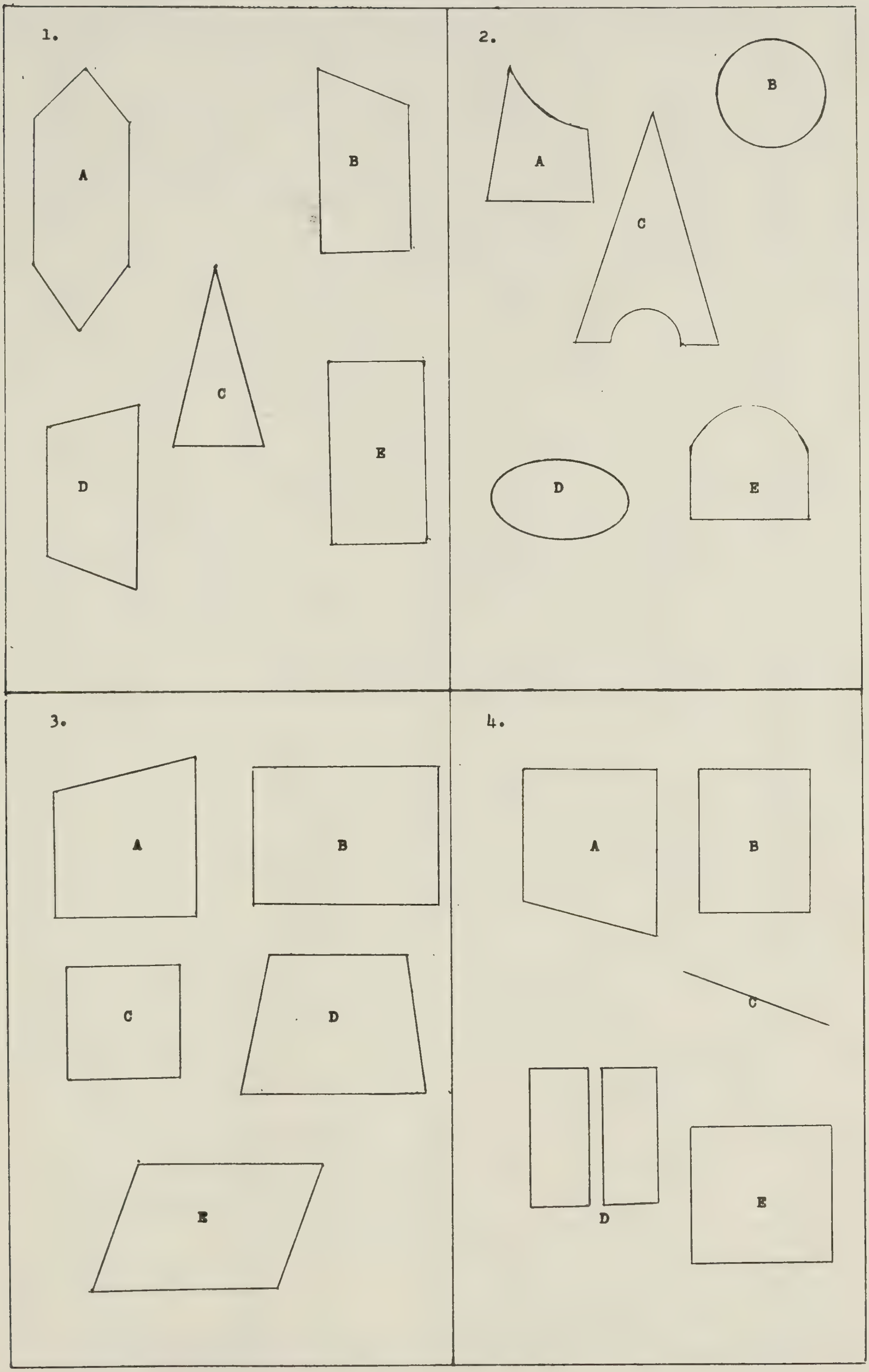
MAKE UP SOME MORE

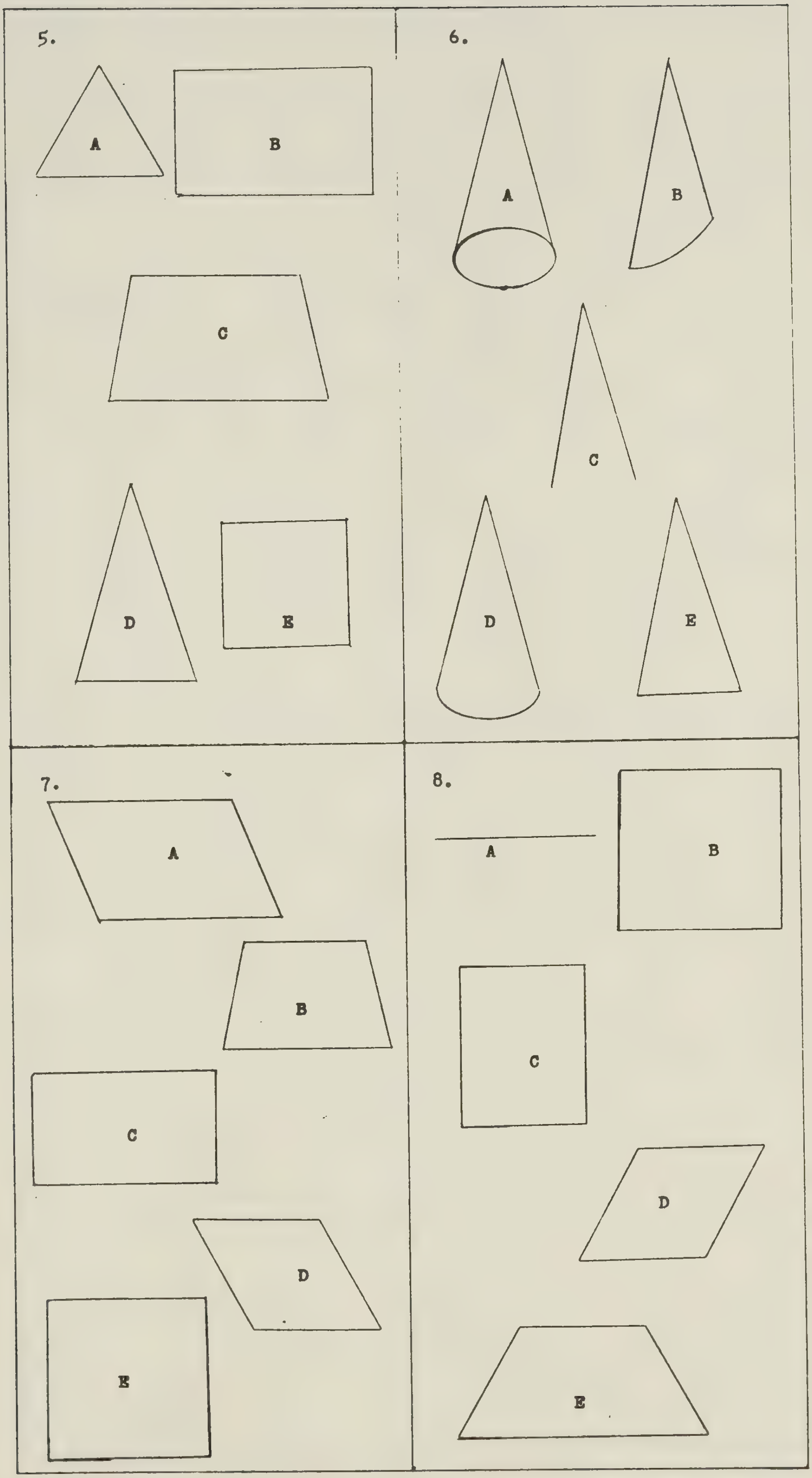
PROBLEMS

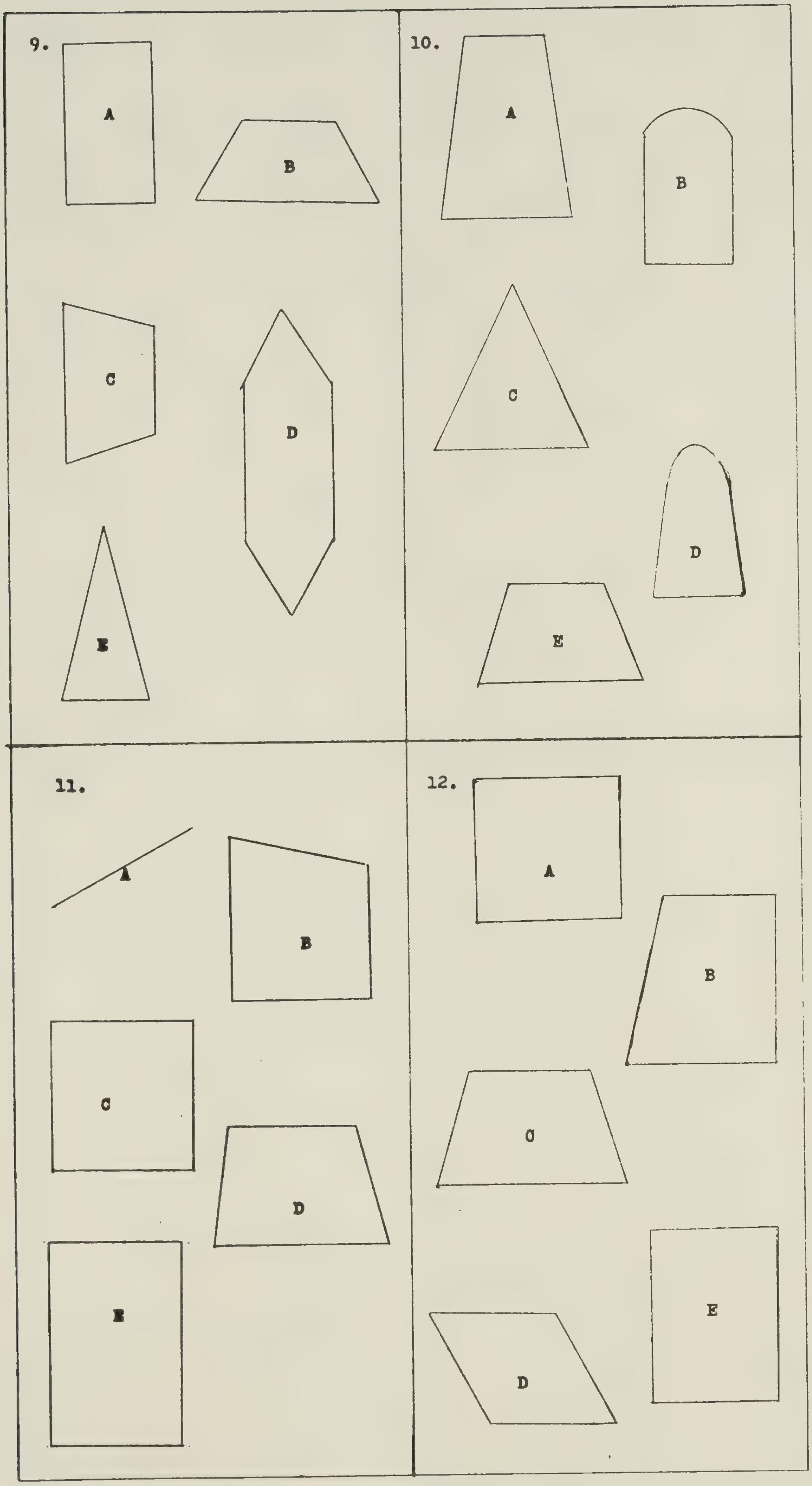
WITH OPEN AND CLOSED

BOXES.

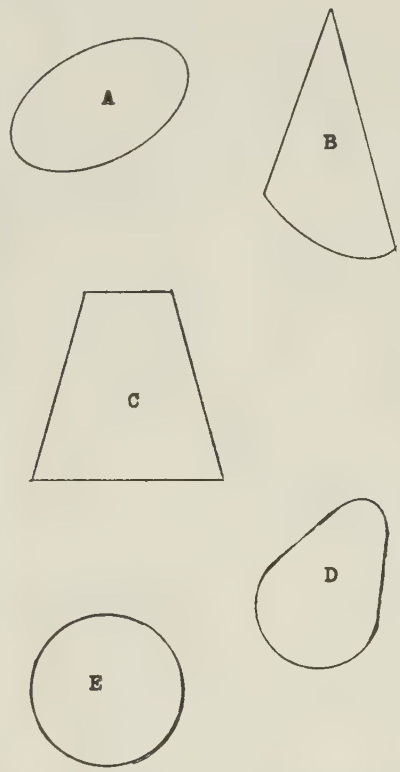
Appendix 2 - Multiple-Choice Pre-Test



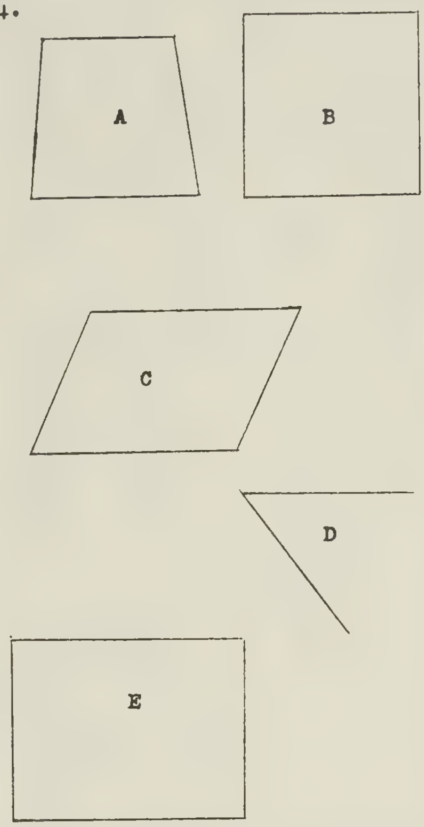

5.
A
B
C
D
E
6.
A
B
C
D
E
7.
A
B
C
D
E
8.
A
B
C
D
E


9.
A
B
C
D
E
10.
A
B
C
D
E
11.
A
B
C
D
E
12.
A
B
C
D
E

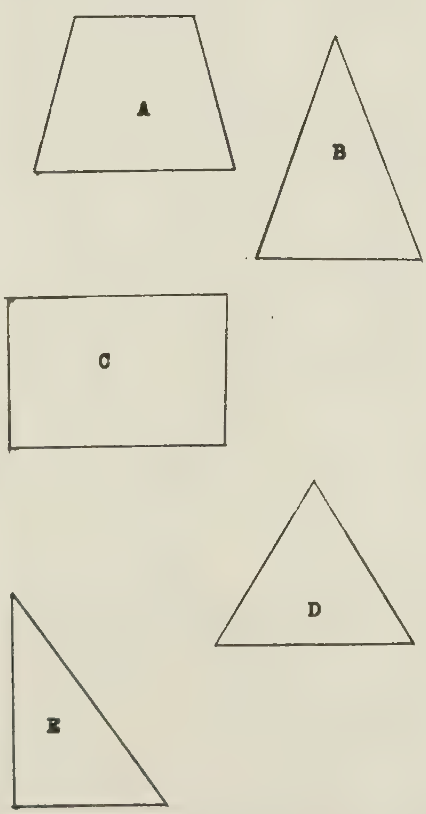
13.



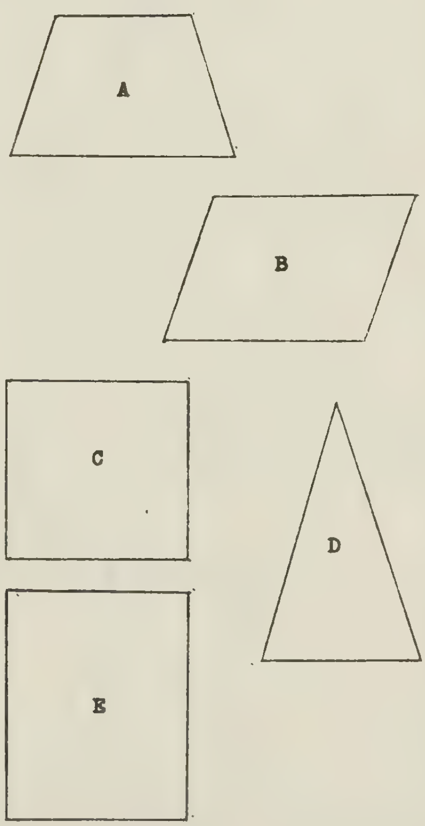
14.



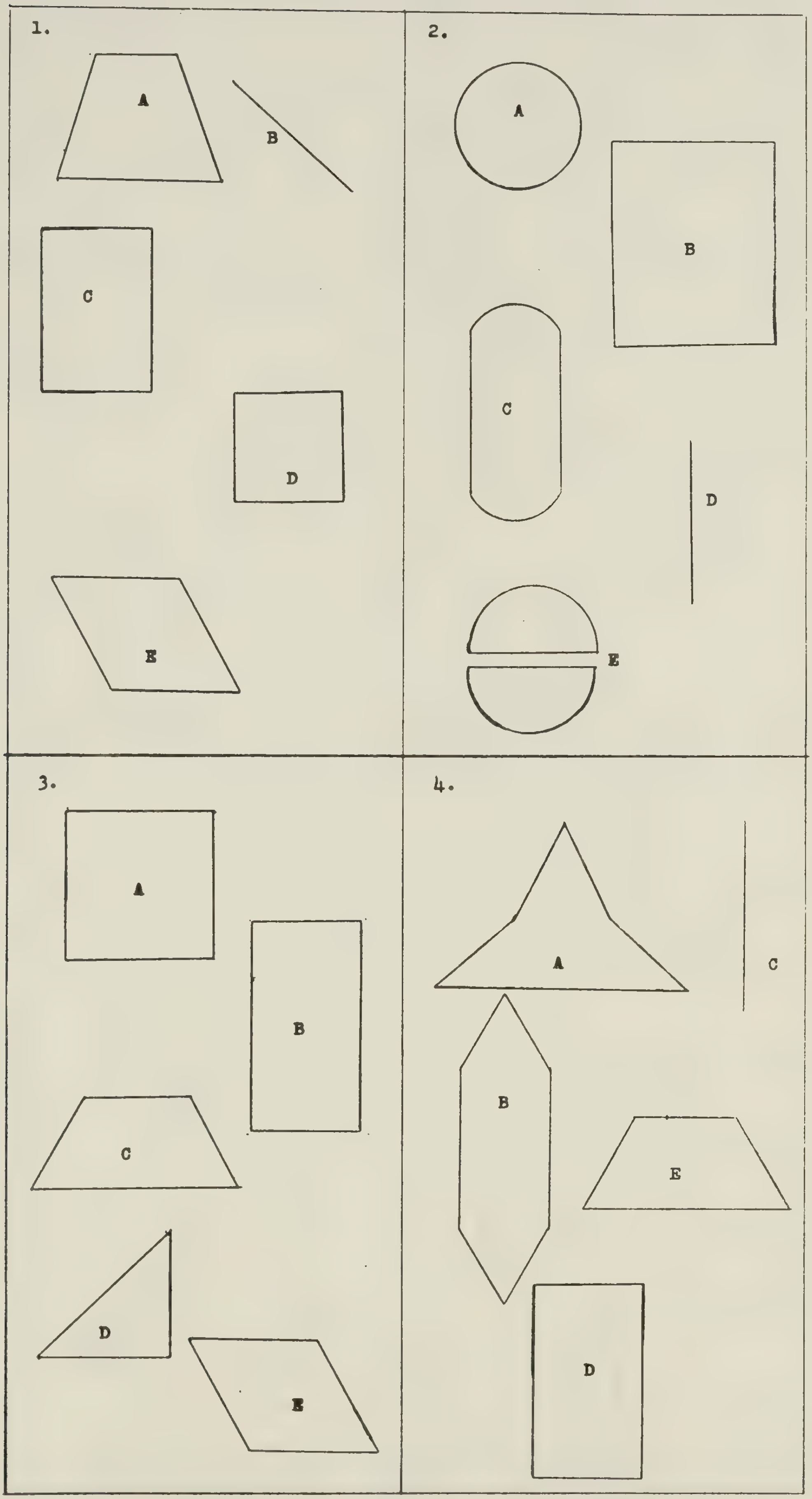
15.

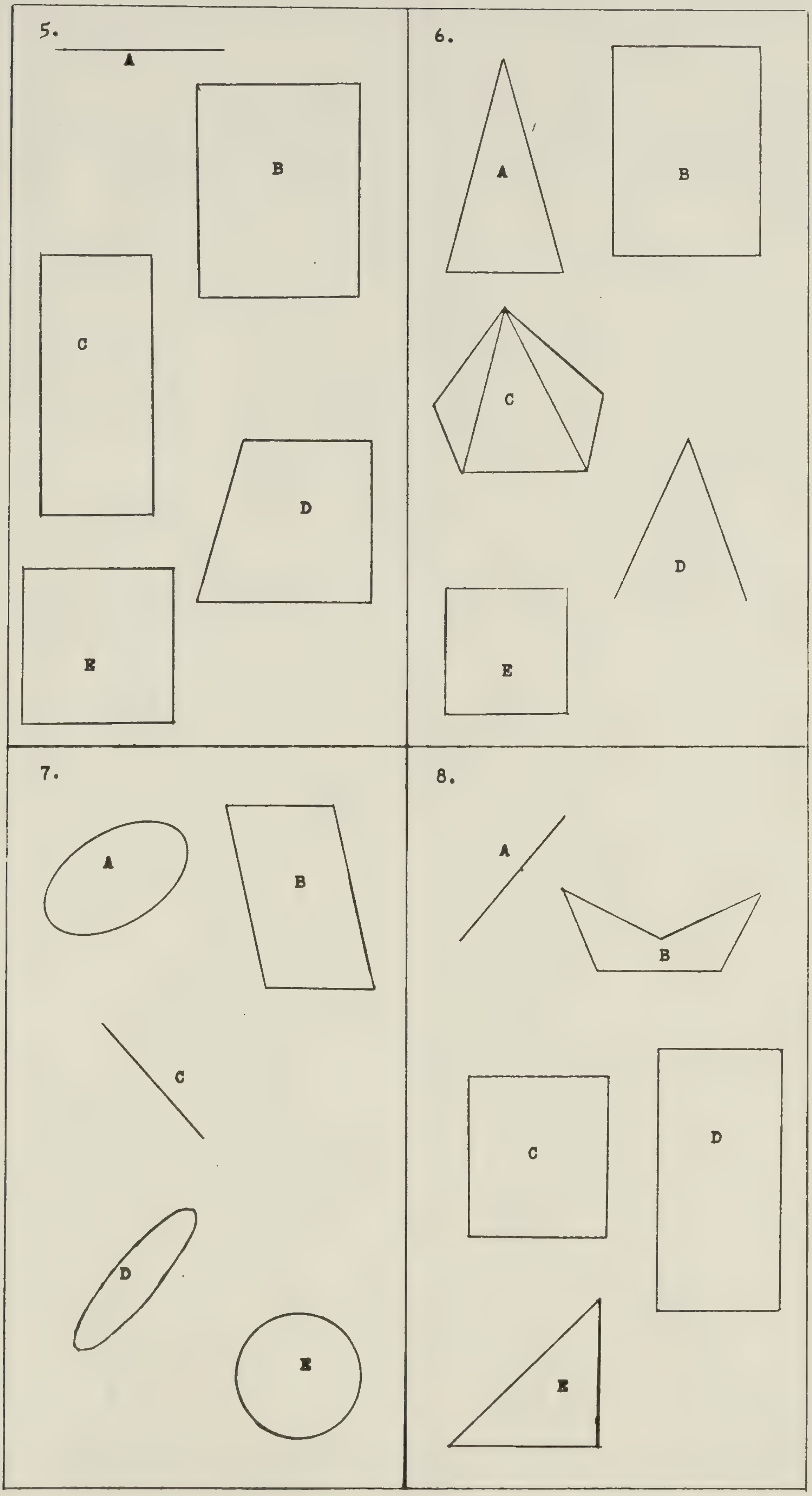


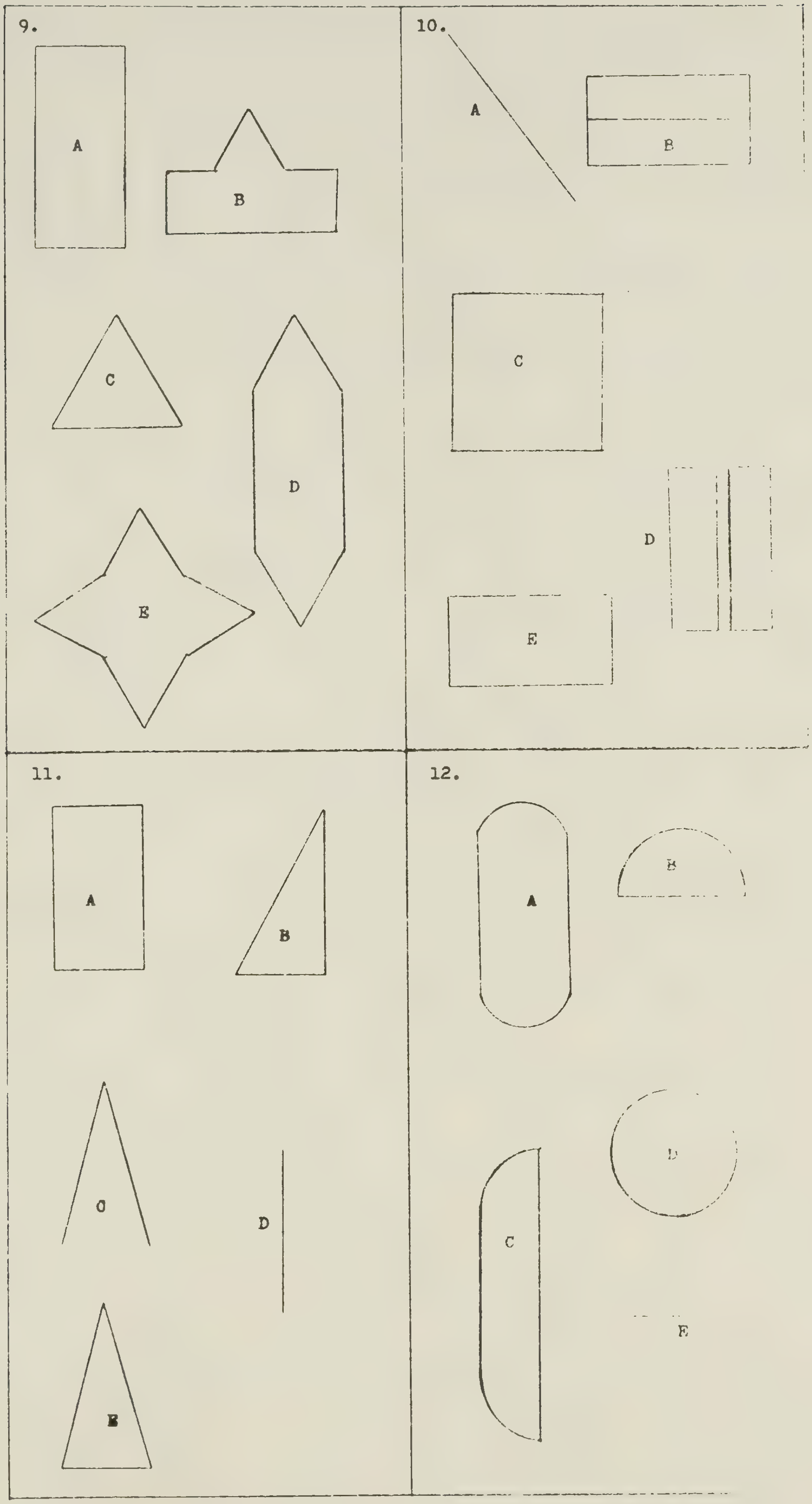
16.

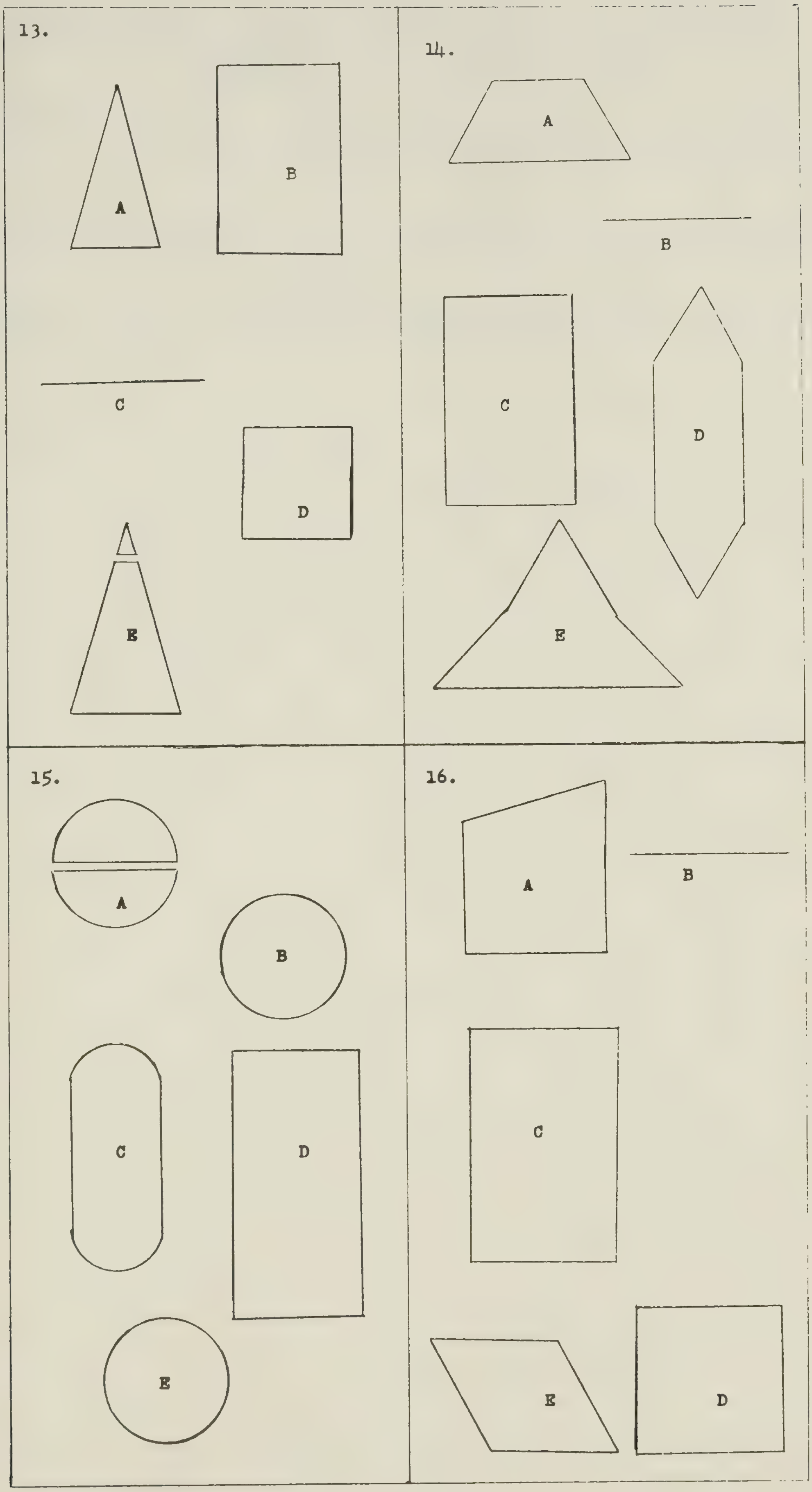


Appendix 3 - Multiple Choice Post-Test




5.
A
B
C
D
E
6.
A
B
C
D
E
7.
A
B
C
D
E
8.
A
B
C
D
E


9.
A
B
C
D
E
10.
A
B
C
D
E
11.
A
B
C
D
E
12.
A
B
C
D
E


13.
A
B
C
D
E
14.
A
B
C
D
E
15.
A
B
C
D
E
16.
A
B
C
D
E

Appendix 4

Record Sheet in Multiple-choice Tests in Haifa

(English Translation Added)

(Name) ________________________________ שם

(School) ________________________________ בית ספר

(Date) ______________ תאריך (Class) ______________________ כתה

ו	ה	ד	ג	ב	א	.1
ו	ה	ד	ג	ב	א	.2
ו	ה	ד	ג	ב	א	.3
ו	ה	ד	ג	ב	א	.4
ו	ה	ד	ג	ב	א	.5
ו	ה	ד	ג	ב	א	.6
ו	ה	ד	ג	ב	א	.7
ו	ה	ד	ג	ב	א	.8
ו	ה	ד	ג	ב	א	.9
ו	ה	ד	ג	ב	א	.10
ו	ה	ד	ג	ב	א	.11
ו	ה	ד	ג	ב	א	.12
ו	ה	ד	ג	ב	א	.13
ו	ה	ד	ג	ב	א	.14
ו	ה	ד	ג	ב	א	.15
ו	ה	ד	ג	ב	א	.16
(F	E	D	C	B	A)	

Appendix 5

Record Sheet for Study "Making Solids from Rectangles"

Marie Kuper

Name of Student .... No ....

School .... Sex ..... Age .... Years ..... Months

IQ .... Grade ....

PRE-TEST

Drawing	Multiple-choice
(Mark either correct..1 or incorrect ....0)	(Circle the student's choice in the same order.)
1.	1. A B C D E F
2.	2. A B C D E F
3.	3. A B C D E F
4.	4. A B C D E F
5.	5. A B C D E F
6.	6. A B C D E F
7.	7. A B C D E F
8.	8. A B C D E F
9.	9. A B C D E F
10.	10. A B C D E F
11.	11. A B C D E F
12	12. A B C D E F
13	13. A B C D E F
14.	14. A B C D E F
15.	15 A B C D E F
16.	16. A B C D E F

(continued)

POST - TEST

Drawing	Multiple-choice	Questions - Part III
1.	1. A B C D E F	1. (a)
2.	2. A B C D E F	1. (b)
3.	3. A B C D E F	1. (c)
4.	4. A B C D E F	1. (d)
5.	5. A B C D E F	1. (e)
6.	6. A B C D E F	1. (f)
7.	7. A B C D E F	1. (g)
8.	8. A B C D E F	1. (h)
9.	9. A B C D E F	2. (1)
10.	10. A B C D E F	2. (2)
11.	11. A B C D E F	2. (3)
12.	12. A B C D E F	3.
13.	13. A B C D E F	4. (1)
14.	14. A B C D E F	2. (2)
15.	15. A B C D E F	5. (a)
16.	16. A B C D E F	5. (b)
		5. (c)

COMMENTS ....

....

....

These questions are scored:

Correct 1

Incorrect ... 0

Appendix 6

Protocol for Administration of the Pre-test

(Version used in Israel, after translation from Hebrew)

1. Distribute part 1. This consists of a booklet containing a name sheet, and ten pieces of plain paper.

2. Ask the class to write name, school, class, birthdate and sex on the front sheet. Ask them to fold each sheet length-wise to form twenty pages.

3. Remove the sphere from the box of models.

"Look at this solid. What is it?"

Discuss the replies with the class. Make sure the children see the difference between a solid body and a plane figure.

"If I cut the sphere with this knife (or ruler representing the knife) what plane shape will we see if we open the ball? The knife goes in here ..... and comes out here."

Discuss this point with the children. Make certain that all the class see the point of entry and exit of the knife. This action should be stressed in every demonstration. Open the ball and let the class see the two cut surfaces.

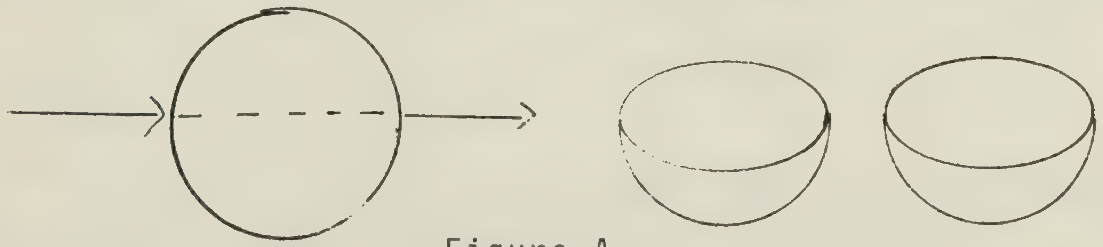


Figure A
Cut Ball

"I know we see two cut halves and two 'caps'. The flat surface is a circle. We see two circles, but they are both the same. Why? (Demonstrate by putting the two halves together again.) Just draw one circle."

Draw a circle free-hand on the blackboard. The children now draw a circle on half of the first page of their book, and label[1] the drawing A.

4. Remove the octahedron from the box. Discuss the solid with the class.

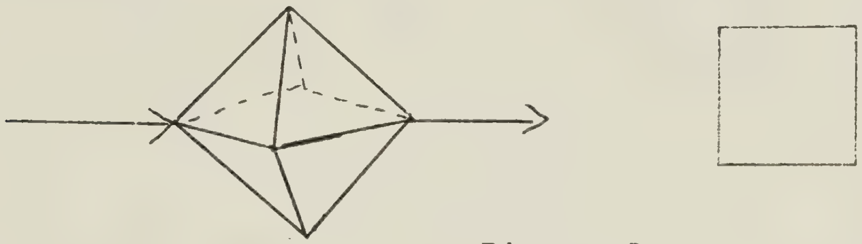


Figure B

The Octahedron

Give its name, and discuss the number of faces, vertices and edges which this solid has. Make sure the children understand these terms.

"I am going to cut this solid here ..... (Demonstrate the transverse cut.) The knife goes in here and comes out here. (Repeat the action several times showing to all sides of the class.) What shape will I see if I open the octahedron?"

After obtaining replies from the children, open the octahedron and show the class the cut surface.

---

[1]Latin letters are used here for labelling, but for the tests in Israel, Hebrew letters were used.

Draw a square on the blackboard, and ask the children to draw a square on the second side of the first page of their book. Label this drawing B.

5. Repeat this procedure for the longitudinal cut of the octahedron (the rhombus) and for the skew cut of the octahedron (the kite).

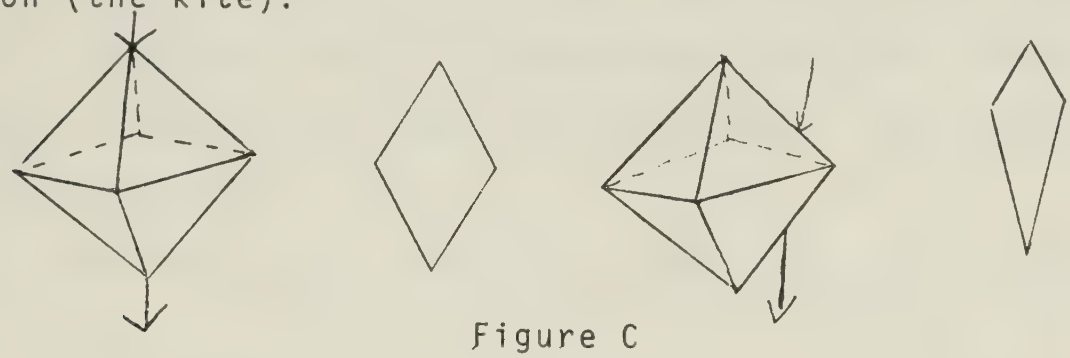


Figure C

The Octahedron

These drawings are copied by the children on the second sheet of their answer books (Drawings C and D). These procedures form the first section of the test and are designed to make sure that the child understands what is required of him.

6. Remove the models used in the trial period, leaving the four test models, namely, the cube, the triangular prism, the cone and the paralellepiped, in the box.

> "I am going to show these models and demonstrate the cuts. This time I will not open the models. You must imagine what the cut surface looks like and draw it. There are two parts each with the same cut surface. We will draw one of them. Draw them in the pages shown. Label your drawings 1. to 16. If you want to see any cut repeated please ask."

7. The solids are to be held the same way for every section. The axis of the solid is to be perpendicular to the floor (Figures 10 and 11). The order of presentation of the solids is shown on the accompanying chart. When not in use the solids are to be kept in the large box, so that they cannot be seen by the class. Repeat the cuts so that every child sees the cut from both the left and right. If a child requests to repeat the cut, do so. Say carefully for each cut:

"The knife goes in here and comes out here. Like this ...."

Remind the children to check that the number of the drawing is correct.

8. The four cuts for each solid are:

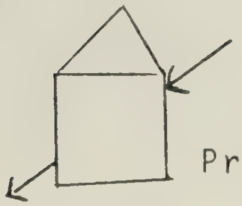
(a) A cut perpendicular to the floor and to the body of the tester - *Longitudinal A.*

(b) A cut perpendicular to the floor and parallel to the body of the tester - *Longitudinal B.* For the cone this is displaced to one side to obtain the hyperbola.

(c) A cut parallel to the floor, from right to left - *Transverse.*

(d) A cut from the top right to the bottom left - *Oblique.* This cut is used in the case of the cube and the parallelepiped. For the triangular prism and the cone the knife enters near the top right edge and emerges on the left edges, near the base.

(The cuts are shown in figures 10 and 11.)



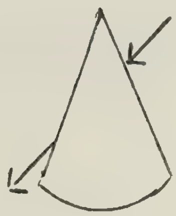


Figure D
Oblique Cuts

9. Check after every cut that the child has drawn the section in the correct space, by repeating the number of the section frequently.

10. After all 16 cuts have been demonstrated and drawn, collect the booklets and distribute part II. This consists of an answer sheet (Appendix 4) and of a book containing 16 pages. Each page consists of one set of five drawings appropriate to the cut demonstrated. (These drawings are shown in Appendix 2.) The procedure is demonstrated to the children on the blackboard.

> "On the answer sheet are rows of six letters. After each cut has been demonstrated choose the diagram which you think is appropriate for that particular section. Ring the letter corresponding to the drawing. If you think that none of the drawings is correct then circle the letter F. Check that the book of drawings is open at the correct page."

It is important to check that every child has the book of drawings open at the correct page before the section is demonstrated. It is also useful to use a ruler or piece of paper to ensure the correct row of letters is being used.

11. The results of the two tests are recorded on a record sheet (Appendix 5). The replies are graded 1 if correct, and 0 if not correct.

Appendix 7

Protocol for Administration of the Post-test

1. The procedure for this test is similar to the pre-test.
2. The children are reminded of the procedure for the pre-test. The trial models (the sphere and octahedron) are shown and the cuts are demonstrated. The sections are drawn on the blackboard, but not by the children.
3. The cuts are demonstrated and drawn as in the pre-test*. The oblique cut for the cylinder is as shown below. For the star prism this cut enters and exits in the "dip" of the star.

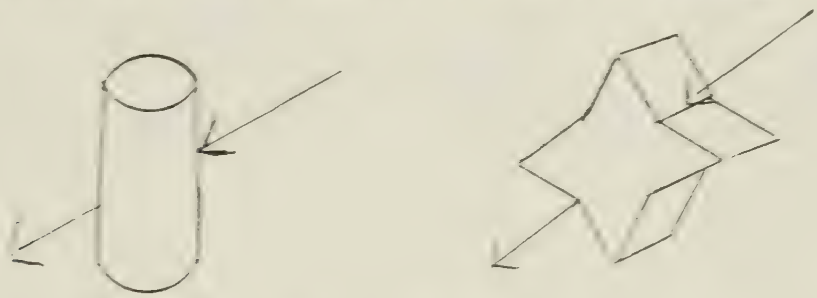


Figure E
Oblique Cuts

4. The multiple choice test is conducted as in the pre-test.
5. The replies are scored 1 for a correct result and 0 for an incorrect result. These scores are entered on the same record sheet as used in the pre-test.

---

*The sections for this test are shown in Figures 12 and 13.

Appendix 8

(English translation included)

משרד החינוך והתרבות

המרכז לתוכניות לימודים

Ministry of Education & Culture

The Curriculum Centre

לשימוש המשרד

For office use

1	2

מספר הכיתה

Class number

3	4

מספר התלמיד

Student's number

5	6

מספר המבחן

Exam number

7	8

נבנה

תיבות

(We build boxes.)

שם התלמיד ______________________

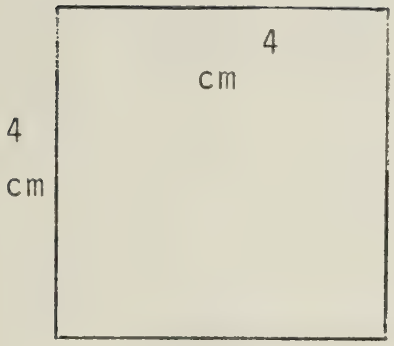
student's name

כתה ______________________

class

בי"ס ______________________

school



1) נתון קטע באורך של 4 ס"מ.

(Given a segment of length 4 cm.)

אפשר לבנות מקטע זה מלבן אחד - הריבוע.

(It is possible to build one rectangle from this segment - the square.)

(זכור: ריבוע הוא מלבן משוכלל)

(Remember - the square is a perfect rectangle.)

(א) כמה מלבנים שונים תוכל לבנות מקטעים שאורכיהם 5 ס"מ ו-או 2 ס"מ? (אפשר להשתמש בכל קטע מספר פעמים).

((a) How many different rectangles can you make using segments whose lengths are 5 cm and 2 cm? (You can use each segment more than once.))

(ב) שרטט את המלבנים.

((b) Draw the rectangles.)

אין צורך בשרטוט מדויק

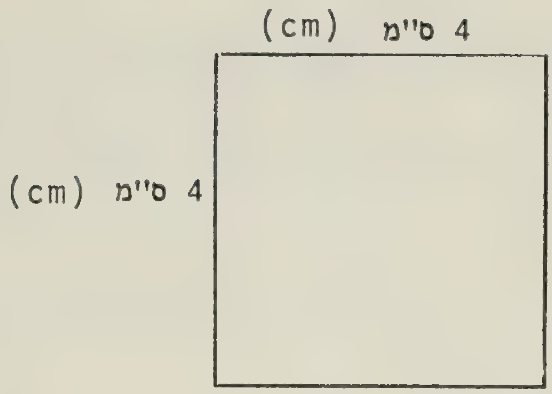
(There is no need to be accurate.)

סמן את אורכי הצלעות בשרטוטיך.

(Mark the lengths of the sides on your drawings.)

2) מריבועים כמו בשרטוט

(From squares like those in the drawing)

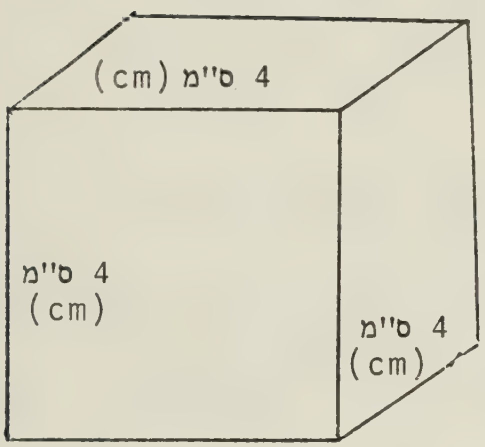


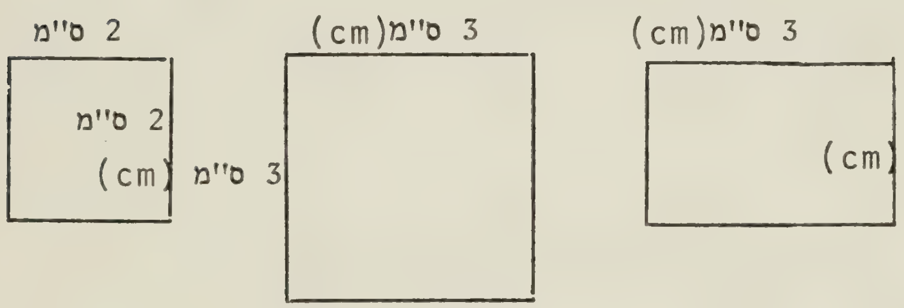
אפשר לבנות רק תיבה אחת - הקוביה

(It is possible to build only one box - the cube.)

בקוביה כל שש הפאות הן ריבועים.

(In the cube there are six faces - they are squares.)





נתונים המלבנים

(Given the rectangles -)

(א) כמה תיבות שונות תוכל לבנות מהמלבנים האלה?

((a) How many different boxes can you make from these rectangles?)

(אפשר להשתמש בכל מלבן מספר פעמים).

(It is possible to use each rectangle a number of times.)

(ב) רשום את מימדי התיבות. התיבות הן:

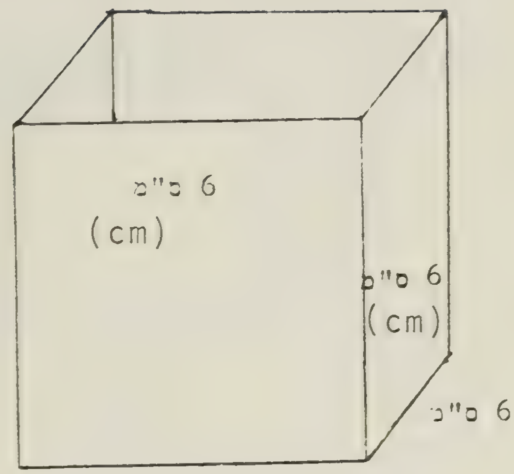
((b) Record the measurements of the boxes. The boxes are:)

3) נתונים 2 תיבות (א) ו- (ב)
אורכי הצלעות מסומנים בציור.

(Given 2 boxes (a) and (b).

The lengths of the edges

are marked on the drawing.)



(a)

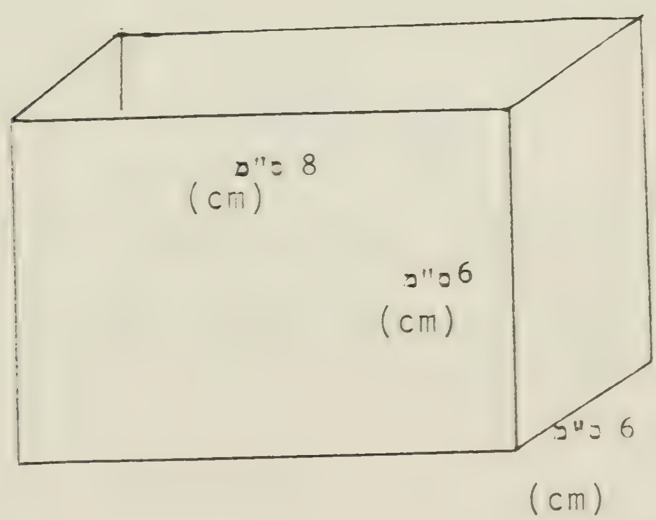
(א) אם הינו ממלאים את התיבות בחול, איזו תיבה היתה מכילה יותר (א) או (ב)?

((a) If we fill the boxes with sand, which box will hold the most (a) or (b)?)

(ב) הסבר (b) (Explain.) ______________________________

____

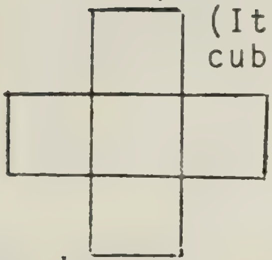
____

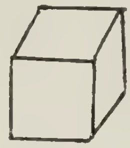


(b)

4) לפניך בצד שמאל יש מעטפת של קוביה פתוחה.
(On the left is a net of an open cube.)

אפשר לקפל אותה ולקבל את הקוביה הפתוחה שמצוירת בצד ימין.
(It is possible to fold the net and get the cube shown on the right.)



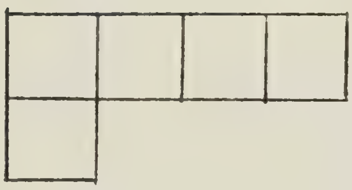


(א) התבונן במעטפות שלפניך. האם אפשר לקפל את המעטפות כך שתתקבל קוביה פתוחה?

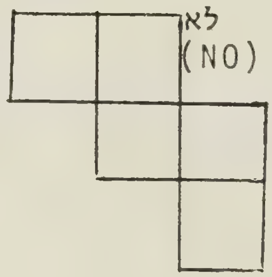
((a) Look at the nets below. Is it possible to fold them to obtain an open cube?)

סמן בעגול את תשובותיך.

(Circle your reply.)



(i) כן (YES) לא (NO)



(ii) כן (YES) לא (NO)

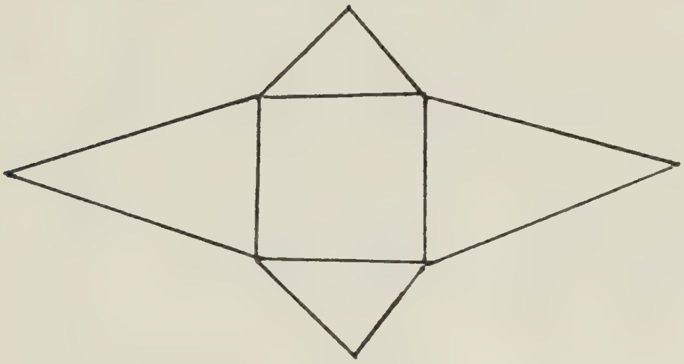
(iii) כן (YES) לא (NO)

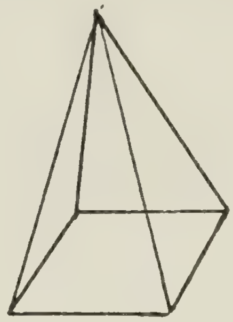
(ב) האם תוכל לשרטט מעטפת שונה מאלה המשורטטות למעלה, שאפשר לקפל אותה ולקבל קוביה פתוחה

((b) Can you draw different nets from those shown above, which can be folded to obtain an open cube?)

5) האם אפשר לקפל את המעטפות שבציור בצד שמאל כדי לקבל את הגופים המצוירים בצד ימין? סמן תשובותיך?

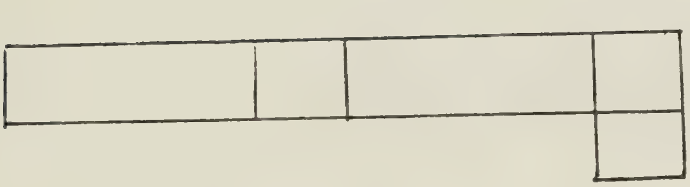
(Is it possible to fold the net drawn on the left to obtain the solid drawn on the right? Circle your reply.)

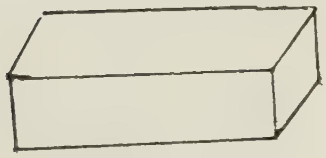




(א) כן לא

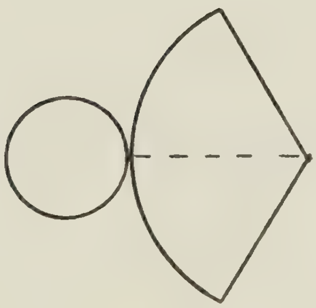
(a) (NO) (YES)

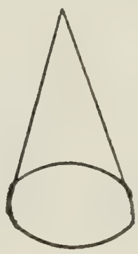




(ב) כן לא

(b) (NO) (YES)



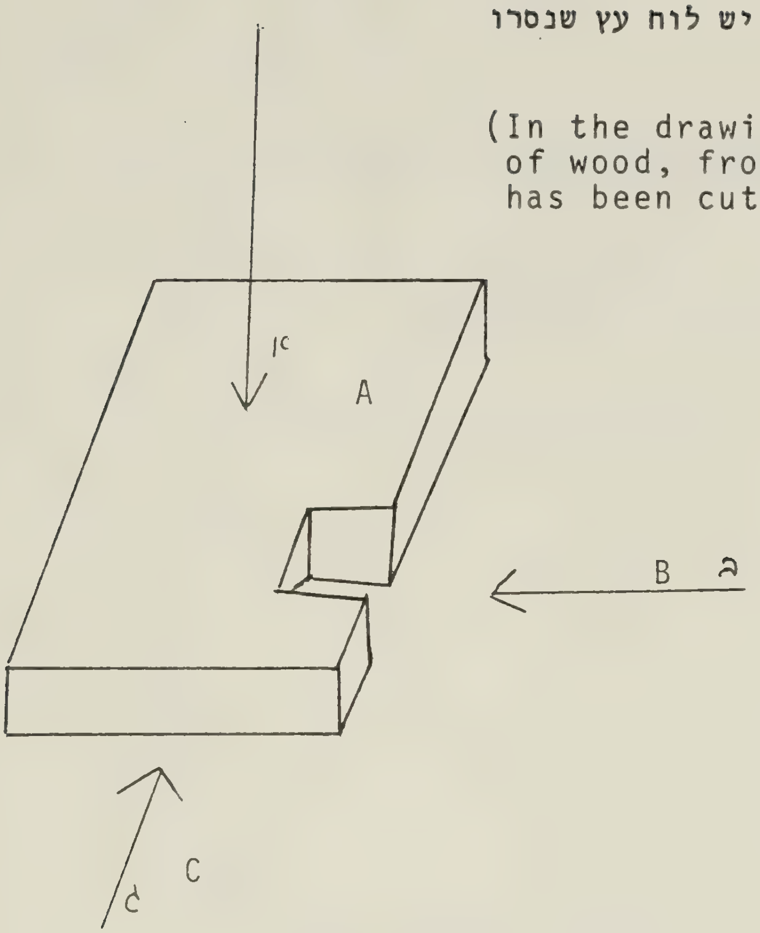


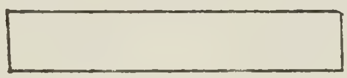
(ג) כן לא

(c) (NO) (YES)

6) בציור שלפניך יש לוח עץ שנסרו ממנו חלק.

(In the drawing is a block of wood, from which a piece has been cut.)





(i) מאיזה כוון ייראה הבול כך ?

(From which direction will the block look like this? Circle your reply.)

סמן את תשובתך

א) מלמעלה — ב) מצד ימין — ג) מלפנים

(A - from above) — (B - from the right) — (C - from the front)

 ?

(ii) מאיזה כוון ייראה הבול כך

(From which direction will the block look like this?)

(א) — (ב) — (ג)

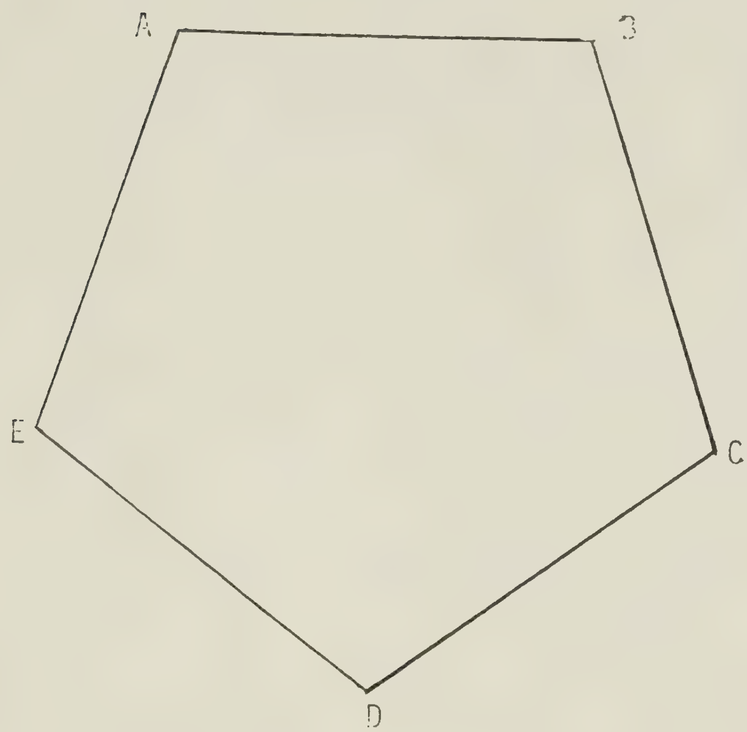
(A) — (B) — (C)

7) הצורה שלפניך הוא מחומש משוכלל. יש לה

(The shape in the drawing is a regular pentagon.)

יש לה 5 צלעות ו- 5 קודקודים.

(It has 5 sides and 5 vertices.)



נניח שיש נקודה P באוויר - בגובה 5 ס"מ מעל פני הנייר. נחבר כל אחד

מהקודקודים A, B, C, D, E עם הנקודה P ונקבל גוף.

(Imagine that there is a point P in the air - 5 cm above the page. Join each of the vertices A, B, C, D, and E, to P and obtain a three dimensional body.)

(א) כמה קודקודים יהיו לגוף שבנית?

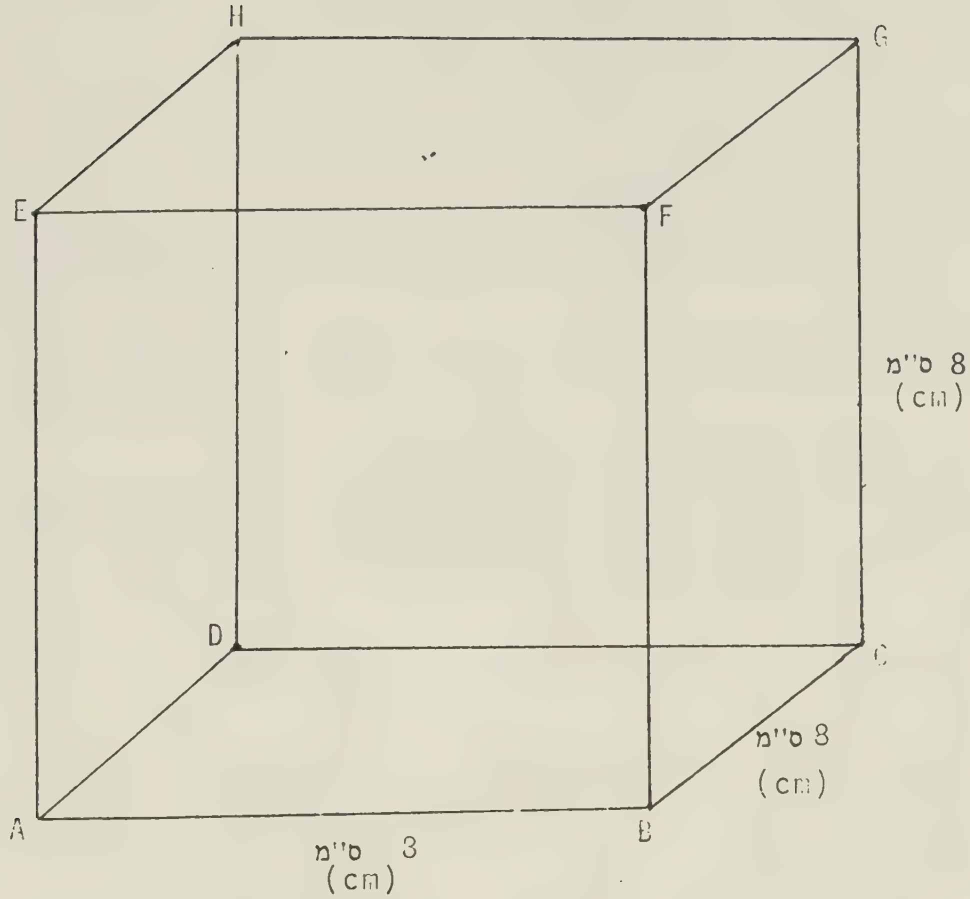
((a) How many vertices does the solid have?)

(ב) כמה צלעות יהיו לגוף שבנית?

((b) How many edges does the solid have?)

(ג) כמה פיאות יהיו לגוף שבנית?

((c) How many faces does the body have?)



לפניך קוביה.
(Here is a cube.)

אורך כל אחת מהצלעות של הקוביה 8 ס"מ.
(The length of each edge of the cube is 8 cm.)

סמן בעגול איזה קטע ארוך יותר בכל אחד מזוגות הקטעים הבאים:
(Circle the segment which is longest in each of the following pairs:)

(א) AG או EB?
(a) or

(ב) HB או EF?
(b) or

(ג) DC או DB?
(c) or

(ד) AC או AG?
(d) or

(9) - הקוביה שבציור היא קובית משחק.

(The cube in the drawing is a die.)

- לא כתובים מספרים על הקוביה.

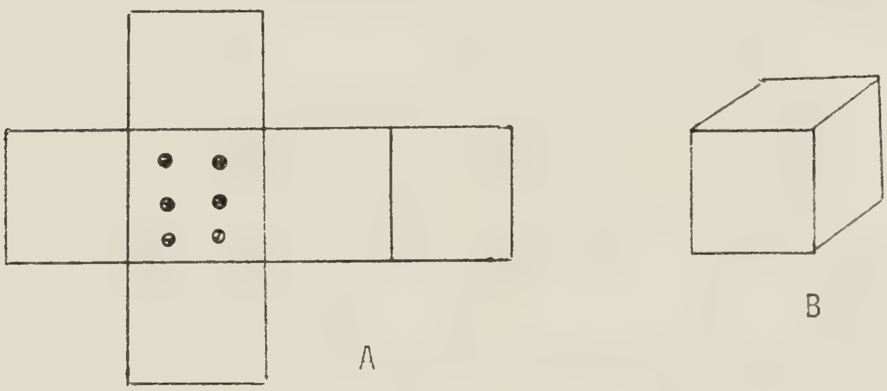
(No numbers are written on the die.)

- בכל קובית משחק, סכום המספרים הכתובים על הפיאות הנגדיות הוא 7.

(On every die, the sum of the numbers on opposite faces is 7.)

- לדוגמא: אם כתוב 5 על הפיאה העליונה, אז המספר 2 יהיה כתוב על הפיאה התחתונה.

(For example: If 5 is written on the top face, then 2 will be written on the bottom face.)



-בציור A יש מעטפת של קובית משחק. המספר 6 כבר במקומו.

(Diagram A represents a net of the die. The number six is already in place.)

(א) סמן את הנקודות 1,2,3,4,5. במקומות הנכונים בציור.

((a) Put the numbers 1, 2, 3, 4, 5, in the correct places in the diagram.)

(ב) האם אתה יכול להשלים את הקוביה B ?

((b) Can you complete the die in diagram B?)

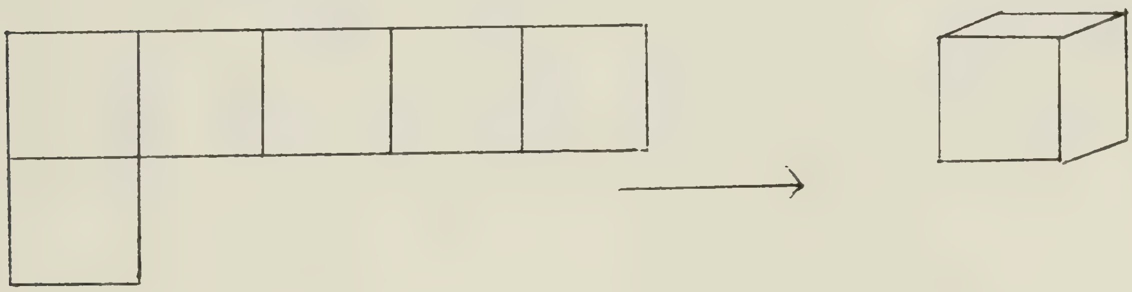
Appendix 9

Achievement Test: Edmonton

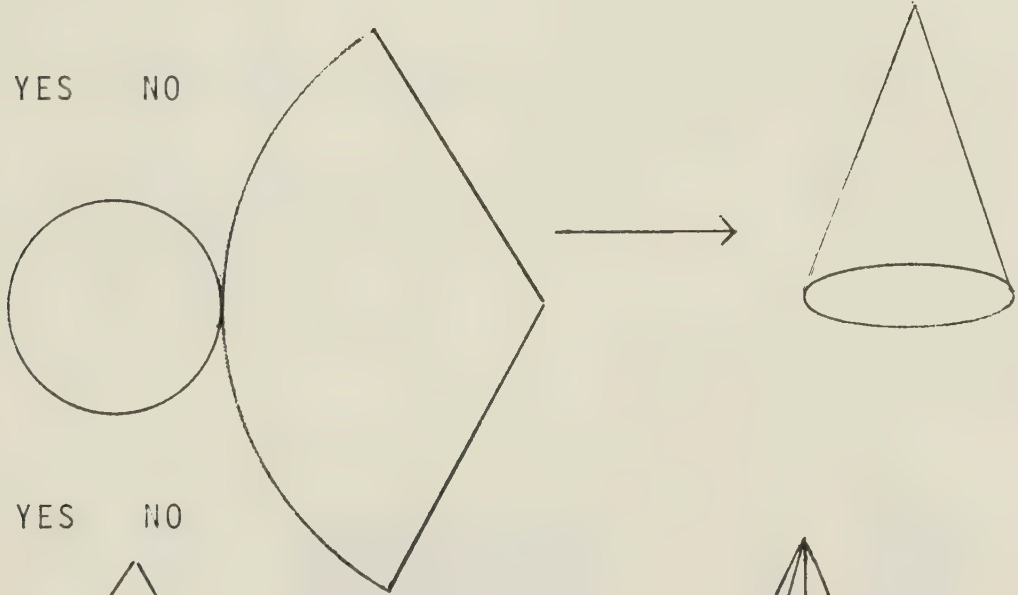
QUESTION 1

The diagrams on this and the next pages show the plans (or nets) of different solid figures. If the plan is cut out and then *folded*, will it make the solid shown on the right? CIRCLE YES OR NO.

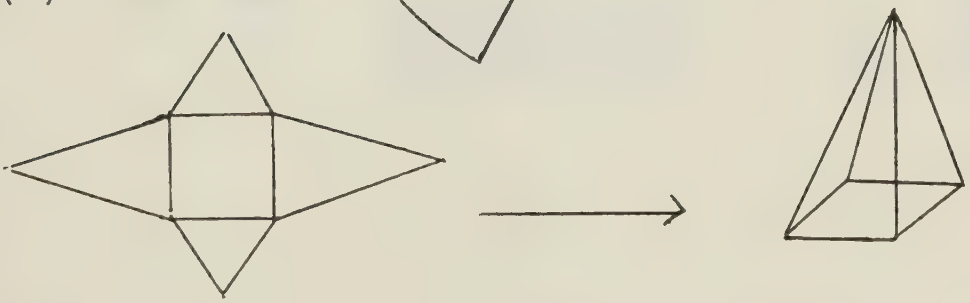
(a) YES NO



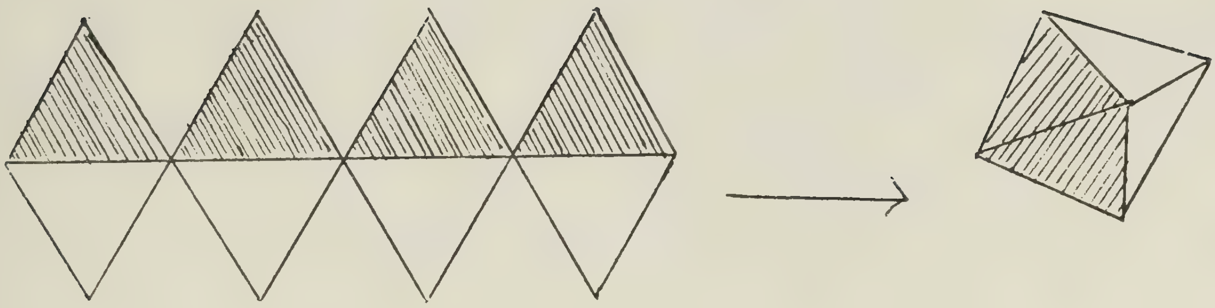
(b) YES NO



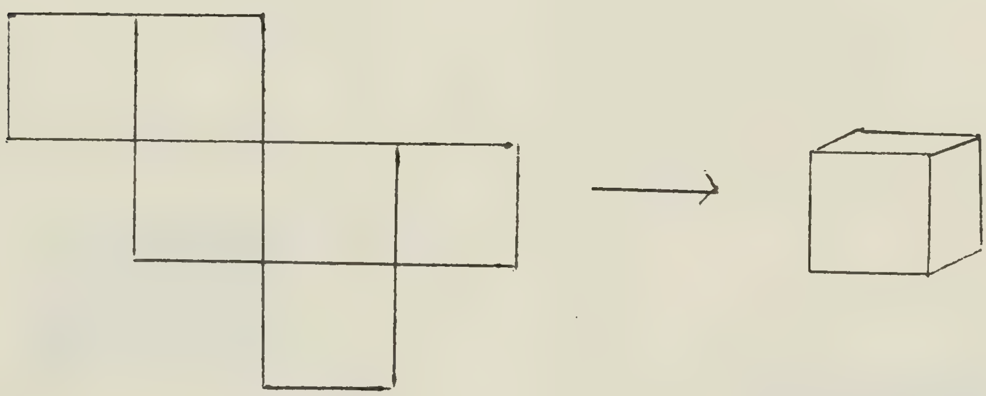
(c) YES NO



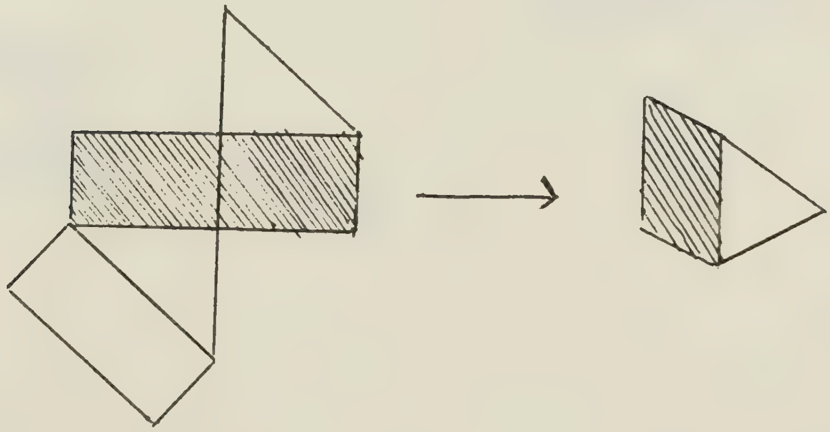
(d) YES NO



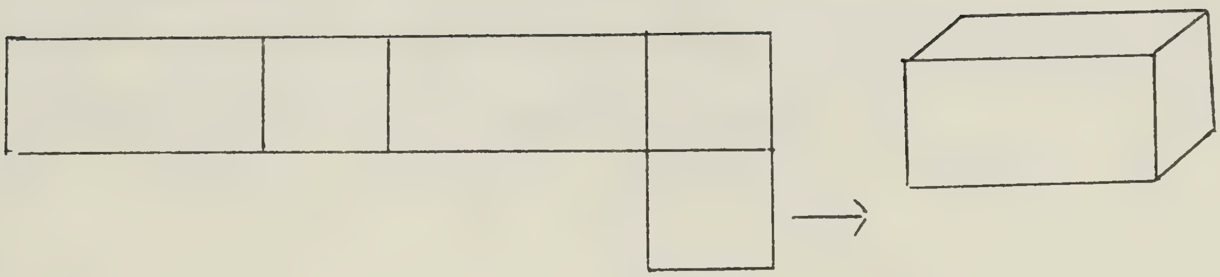
(e) YES NO



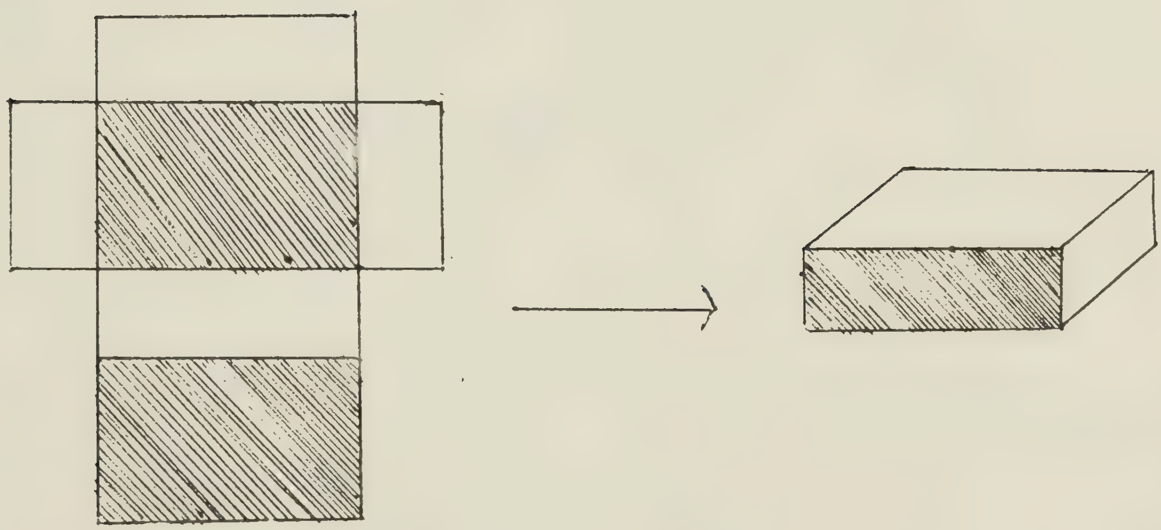
(f) YES NO



(g) YES NO



(h) YES NO

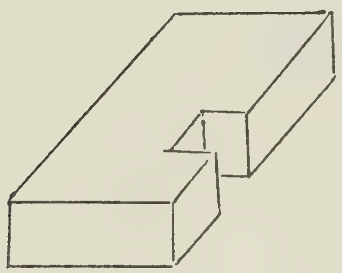


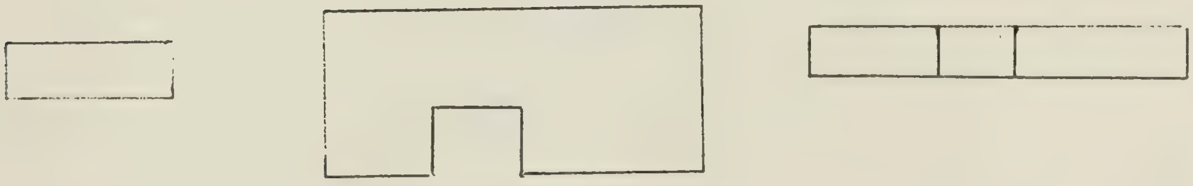
QUESTION 2

The diagram shows a block of wood in which a groove has been cut. (Hidden lines are not shown.)
If you look at the block from different positions, which of the diagrams A, B, or C would represent:

1. The top view A B C
2. The view from the *right* side A B C
3. The front view A B C

CIRCLE THE CORRECT ANSWER.





A B C

QUESTION 3

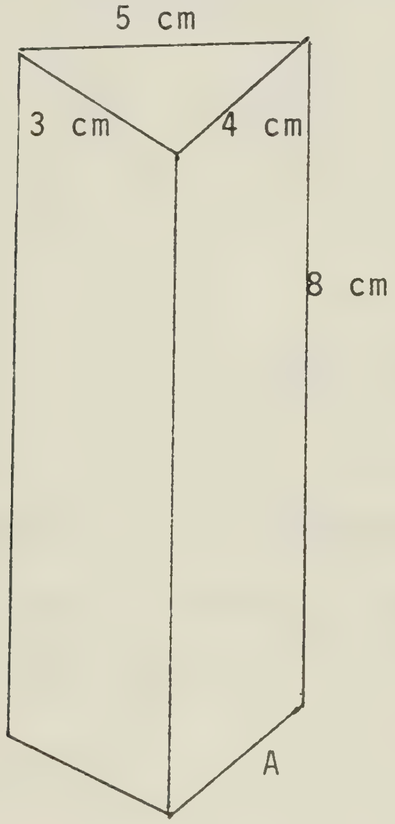
The shape A is a prism with a triangular base.

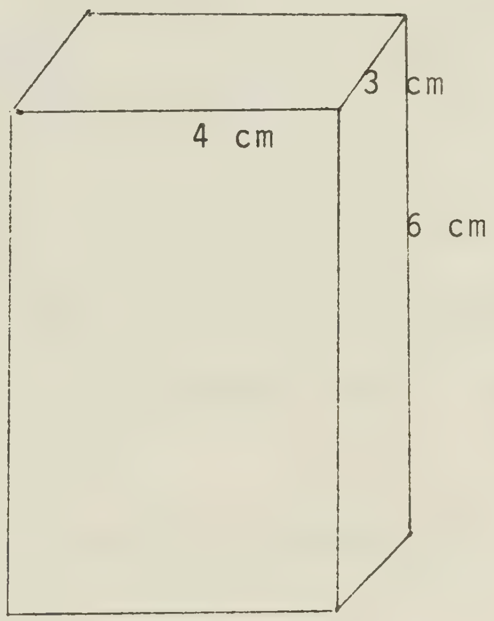
The sides of the base are 3 cm, 4 cm, and 5 cm respectively.

The height of the prism is 8 cm.

Shape B is a rectangular box. The sides of the base are 3 cm and 4 cm. The height of the box is 6 cm.

Which shape is the biggest? Can you explain why?





B

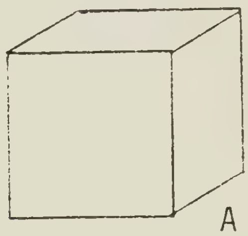
The teacher will show you the prism and the box.

ANSWER

QUESTION 4

The diagram A shows a die (sometimes called a dice) used for playing games. The numbers are missing from the die. Diagram B shows the net (or plan) of the die. It could be cut out, and folded to make the die. The face with the "six" is shown on the net.

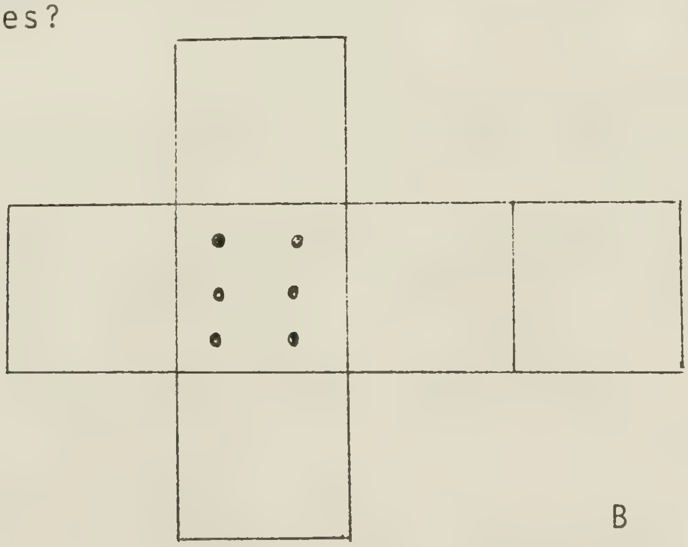
The numbers on the opposite faces of the die always add up to 7. Thus if 5 is on the top, then 2 must be on the bottom.



A

If the six is in the position shown on the net, show how the numbers 1, 2, 3, 4 and 5 could be put on the net to make a proper die. (In diagram B)

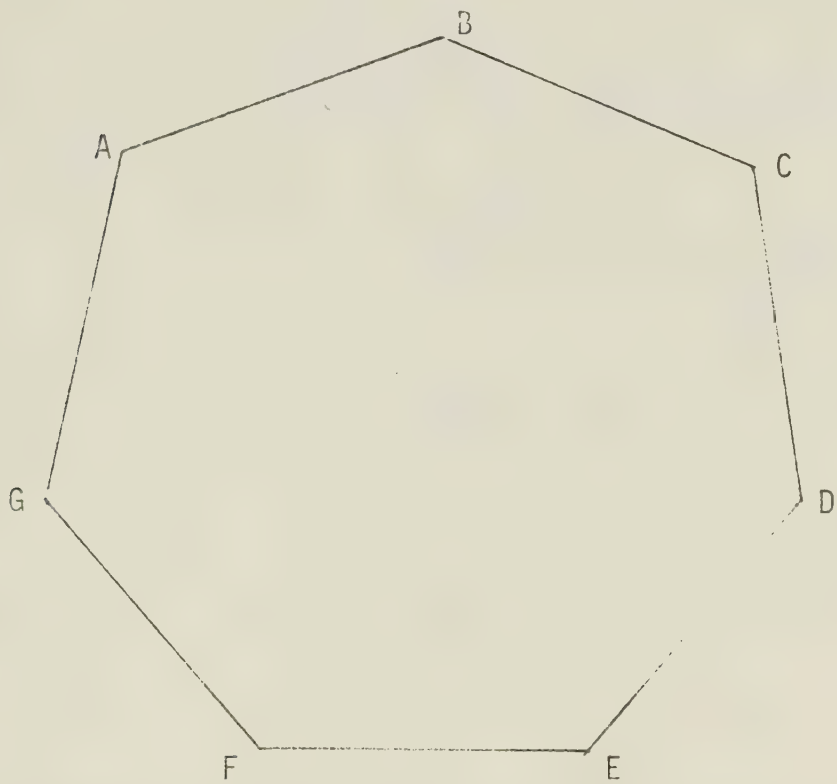
Can you put some of the numbers onto the diagram A in the correct places?



B

QUESTION 5

The figure in the diagram has 7 sides (a heptagon). Imagine a point in the air about 2 inches above the paper. Call this point P. Now try to imagine the solid figure made by joining P to each of the corners A, B, C, D, E, F and G of the heptagon in turn.



(a) How many faces does this solid have? ....

(b) How many edges does this solid have? ....

(c) How many vertices (corners) does this solid have? ......

QUESTION 6 (not used in final test)

Diagram I represents a square room with a square ceiling EFGH and a square floor ABCD. All the sides are 12 feet long. The room is also 12 feet high.

S is a point on the wall ADHE , 2 feet up from the floor AD and 4 feet from the edge EA.
T is a point on the wall BCGF, also 4 feet from the edge FB, and 2 feet down from the ceiling FG.

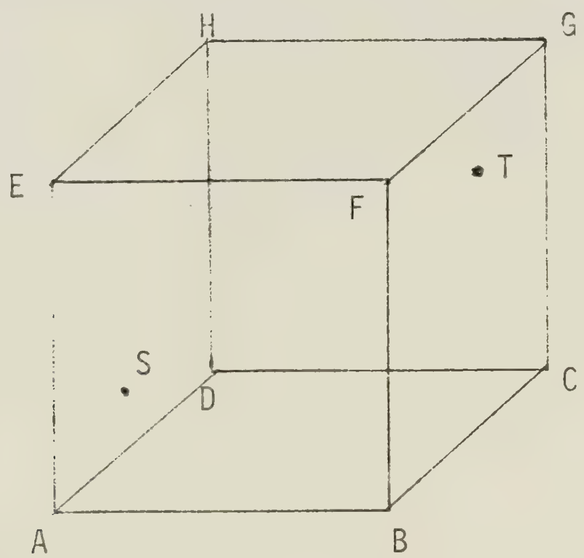


Diagram I

(a) A spider is at S and wants to *crawl* to T. He must crawl along the surfaces of the room. What is his shortest route?

Draw the shortest route on diagram I.

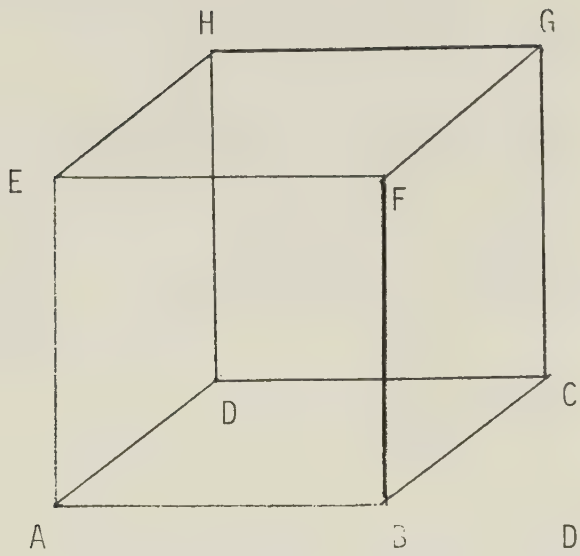


Diagram II

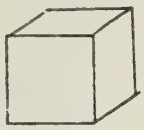
(b) What will be the shortest route if the spider is at A and must get to G? Again he can only crawl on the surfaces of the walls or the ceiling or floor. Draw the route on diagram II.

QUESTION 7 (not used in final test)

(i) The cube in diagram A is a unit cube. All the edges are 1 unit long.

If the cube in the diagram is painted, how many of its *faces* are painted?

(a) ....

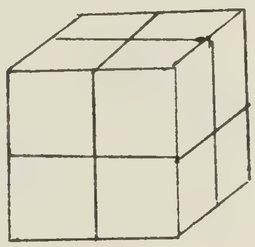


A

(ii) The cube in diagram B is built of 8 unit blocks. Each edge is 2 units long.

If the outside of this cube is painted, how many blocks have

(a) none of faces painted?.........



B

(b) one face painted?.............

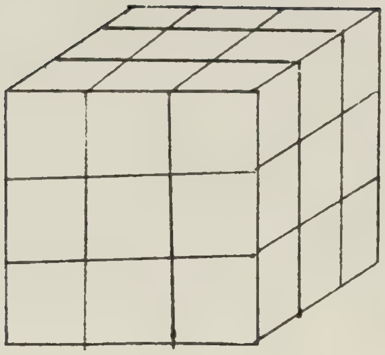
(c) two faces painted?............

(d) three faces painted?..........

(e) four faces painted?...........

(f) five faces painted?...........

(g) six faces painted?............



C

(iii) Diagram C is also built from unit blocks.

Each edge is three units long.

(a) How many unit blocks are there here?............

Now answer questions (b) to (h) for *this* block.

(b) ....

(c) ....

(d) ....

(e) ....

(f) ....

(g) ....

(h) ....

Appendix 10

Teachers' Guide

(Translation of the version used in Israel)

<u>Aims</u>

This unit is designed to serve as an introduction to the study of geometrical solids in the sixth and seventh grades. It was decided to begin with the general rectangular prism, since material on the cube and the regular solids already exists in the text books used in schools. The unit provides opportunities for the child to build three-dimensional models, to compare their sizes, and to identify special lines in relation to the solids. The unit deals with number patterns and simple combinatoric problems. It can be used as an introduction to volume, as revision of area and problems of relationships, and as a basis for finding functions.

The material is suitable for children of various ages and abilities. The children use concrete materials when working through the unit. They can make hypotheses and test them, and solve the problems in an abstract way. It is possible that children of the same age may differ widely in their ability to perform the various assignments in the unit.

The work can be done by a child by himself, or working in groups of three or four.

Page 2 in the unit

The children should be given pieces of stick or drinking straws of lengths 3 cm, 5 cm, and 8 cm and asked to find the number of different rectangles. After the children have had time to think about and experiment with the problem, discuss the results with the class as a whole, to find out what the children discovered and how they reached their conclusions. If the children have had no experience in solving problems of this type, it is possible to guide the work by asking first: "How many rectangles can one make from one type of length? From two? ..."

It may be necessary to remind the students that a square is a special rectangle. The square is a rectangle, but a rectangle does not have to be a square.

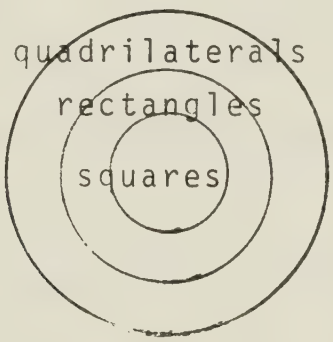


Figure A1

Venn Diagram of Quadrilaterals

With one type of segment, it is possible to make only one rectangle, namely the square. Now get the children to examine the case when there are two segments, say 3 cm and 5 cm. It is possible to use each segment as many times as

necessary, but we do not allow combining segments to make a longer side (e.g. 13 cm from a 5 cm and an 8 cm segment).

Three rectangles can be made from two different segments. For the lengths 5 cm and 8 cm, one can make the squares, 5 cm by 5 cm and 8 cm by 8 cm, and the rectangle 5 cm by 8 cm. The question of whether or not the two rectangles in figure A2 are different should be discussed.

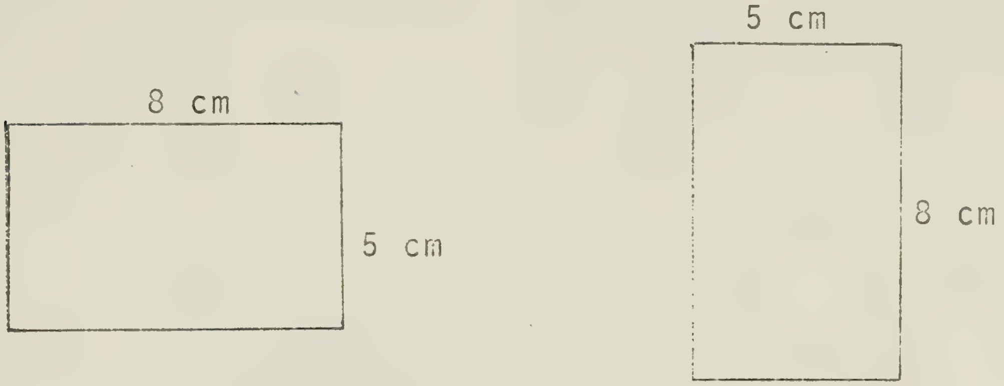


Figure A2

Rectangles

The problem should then be extended to three, four and more segments.

For three segments a, b, and c we have:

Squares: a,a; b,b; c,c .

Rectangles, not squares: a,b ; a,c; b,c .

For four segments a , b , c , d we have the following rectangles (arranged in a different grouping):

a,a	b,b	c,c	d,d
a,b	b,c	c,d	
a,c	b,d		
a,d			

With this arrangement it is easy for the child to generalize the problem to n segments.

Finally if the results are tabulated we have:

Table A1

Rectangles Made From n Segments

Number of segments	Number of rectangles	Number of squares	Number of non-square rectangles
1	1	1	1
2	3	2	1
3	6	3	3
4	10	4	6
5	15	5	10
.	.	.	.
.	.	.	.
.	.	.	.
n	$\frac{n(n+1)}{2}$	n	$\frac{n(n-1)}{2}$

The numbers obtained in the second and fourth columns are the triangular numbers.

One way to obtain this relation is first to calculate the number of non-square rectangles, i.e., $\binom{n}{2} = \frac{n(n-1)}{2}$. To this must be added the number of squares n .

It is also useful here to discuss with the children other problems leading to the same series. For example:

(a) How many handshakes will there be if there are 3, 4 or more people in a room and each shakes

hands with every one in the room?

(b) The number of diagonals in a polygon.

(c) The number of angles formed by n rays meeting at a point.

(d) The number of segments in a line, when n points are marked on the line.

Pages 3 and 4 in the unit

"What type of geometrical solids can you make from the rectangles on page 3? You can use each type of rectangle as many times as you please." Discussion of this question can be used as an opportunity to remind the students of the names of solids, such as prism, pyramid and cone.

Page 5 in the unit

"How many different boxes can you build?" There is no need to explain what is meant by "different". When the children make duplicate boxes it is possible to point out that the two boxes on page 5 are the same box. The children are given more rectangles than they need to complete the assignment, to allow for duplicates to be made.

There is no need at this stage to demand accuracy or neatness in building. There will be opportunities later for this type of activity. If the students find the building difficult, encourage them to make nets first (Figure A3).

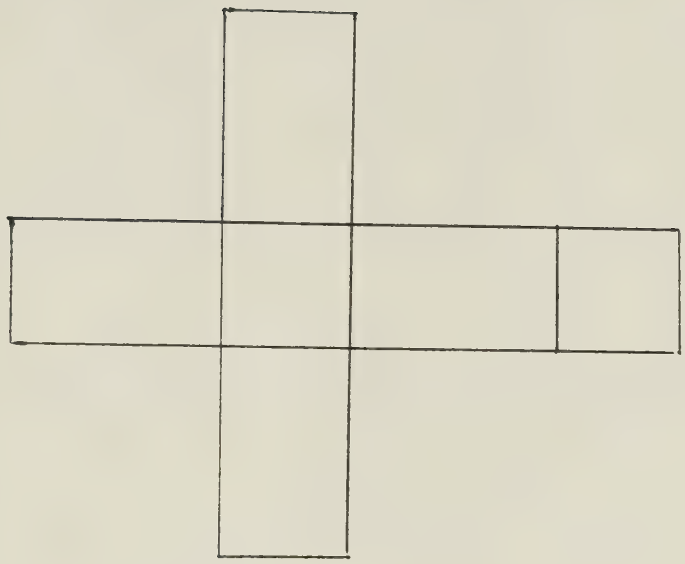


Figure A3

Net of Rectangular Box

Pages 6 and 7 in the unit

It is possible to build 10 different boxes from the rectangles shown on page 3, i.e. of side lengths a, b and c. (In the unit the lengths 3, 5 and 8 cm are used.) The notation 3, 5, 8 is used to designate the box whose side lengths are 3 cm, 5 cm and 8 cm. The notation 3 x 5 x 8 is avoided, as it implies multiplication, and hence volume. However multiplication is only one of the properties of a rectangular prism.

The possibilities are:

Table A2

Rectangular Prisms Made with 3 Different Edge Lengths

(a,a,a)	(3,3,3)	(a,c,c)	(3,8,8)
(a,a,b)	(3,3,5)	(b,b,b)	(5,5,5)
(a,a,c)	(3,3,8)	(b,b,c)	(5,5,8)
(a,b,b)	(3,5,5)	(b,c,c)	(5,8,8)
(a,b,c)	(3,5,8)	(c,c,c)	(8,8,8)

Notice the different ways in which the children arrange the boxes and check to see if they have found all possibilities. They may first check to see if they have made all the cubes. They may measure the sides. If the children find this exercise too difficult they can match their boxes with those in the booklet. The missing boxes on these pages are the three "largest" boxes: (5,5,8), (5,8,8) and (8,8,8).

At this stage the Table A1 of this guide can be extended. It is recommended that this activity is suitable for children of high ability. The extension is shown in Table A3.

Table A3

Rectangles and Rectangular Prisms Made From n Segments

Number of segments	Number of rectangles	Number of boxes
1	1	1
2	3	4
3	6	10
4	10	20
5	15	35
.	.	.
.	.	.
.	.	.
n	$\frac{n(n+1)}{2}$	$\frac{n(n+1)(n+2)}{6}$

To encourage the children to find the "pattern", they should first see that the number of boxes in the m-th row is the sum of the number of rectangles in all the rows up to the m-th row. For example, for 5 segments the number of boxes is 1 + 3 + 6 + 10 + 15 = 35.

To find the total number of boxes, we first compute the number of boxes with no square sides. This is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{1.2.3}$$

For the number of boxes with two square faces we are free to choose only two lengths, that is the length of the side of the square and the length of the third edge. For each choice of a and b we get two different boxes, namely $a \times a \times b$ and $a \times b \times b$.

Thus we must multiply $\binom{n}{2}$ by 2.

There are n solids all of whose faces are squares, i.e. the cubes. Obviously it is not possible to make boxes with four or with an odd number of square sides, so our count is complete. In all we have

$$\frac{n(n - 1)(n - 2)}{1.2.3} + \frac{2n(n - 1)}{1.2} + n$$

$$= \frac{n(n + 1)(n + 2)}{6}$$

different boxes. This result is equal to

$$\sum_{1}^{n} \frac{m(m + 1)}{2}$$

Notice that the number of boxes without square faces is

$$\frac{n(n - 1)(n - 2)}{6} \quad .$$

Notice when $n = 3$

$$\binom{3}{3} = \frac{3.2.1}{1.2.3} = 1 \quad .$$

Thus there is only one box with no square faces.

(Kuper and Walters (1976))

Page 8 in the unit

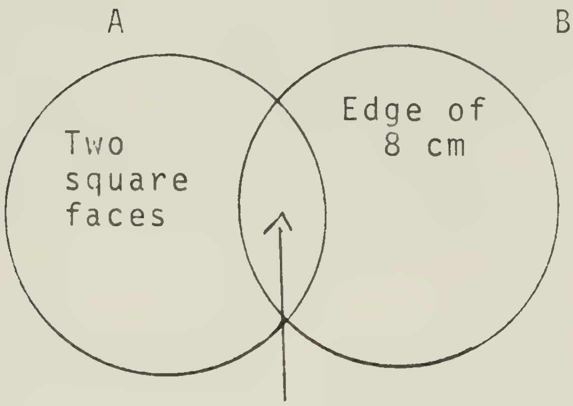
The aim of this activity is to enable the children to recognize the set of boxes, before the subject of their volumes is introduced. The children can sort the boxes in many different ways. Encourage them to sort the boxes into more than two groups. Here are some possible arrangements:

Table A4

<table>
<tr><th colspan="12">Sets of Boxes</th></tr>
<tr><td colspan="12">Two sets</td></tr>
<tr><td colspan="6">A
All those with a 3,3 face
All those with a square face
Those with a 3 (or 5 or 8) cm edge</td><td colspan="6">B
Those with no 3,3 face
Those with no square face
Those without a 3 (or 5 or 8) cm edge</td></tr>
<tr><td colspan="12">Three sets</td></tr>
<tr><td colspan="4">A
6 faces all the same</td><td colspan="4">B
A maximum of 4 faces all the same</td><td colspan="4">C
Maximum of 2 faces the same.</td></tr>
<tr><td colspan="12">Four sets</td></tr>
<tr><td colspan="3">A
All those with square faces 8,8</td><td colspan="3">B
All those with square faces 5,5</td><td colspan="3">C
All those with square faces 3,3</td><td colspan="3">D
No square faces</td></tr>
</table>

The students may be surprised to find that there is only one box in set D, i.e., with no square face.

If the class has learnt some set theory, they could be asked to sort the boxes into two sets with non-zero intersection. For example:



There are boxes that fulfil the condition:
edge of 8 cm and two square faces.

Figure A4

Venn Diagram
for the Box Sorting Game

Page 9 in the unit

It can be quite difficult to find the missing box. The task is made easier if the boxes are sorted in a systematic way.

Page 10 in the unit

This page is used to motivate the introduction of volume.

Page 11 in the unit

Before the children can calculate the volume, it is

necessary to know the algorithm, that the volume of a rectangular prism is the product of the three edge lengths. This motivation is provided by the investigation on the previous page. In order to check the order of the boxes, the students have to calculate or measure the volumes.

The boxes made by the children are usually not strong enough for the work in this section. It is recommended that the teacher provides each group of 4 or 5 children with a set of well-made open boxes. Unit cubes or Cuisenaire rods make a good material with which to fill the boxes. Many children discover for themselves that there is no need to fill the box completely, it is enough to measure the three side lengths and to multiply the three numbers. If the discovery does not come naturally, one can ask the question "What can you do if you haven't enough blocks?" In addition if the table (page 11 of the unit) is filled in on the blackboard, many children will see that for the 3, 5, 8 box the number 120 has been obtained by multiplication.

Note that the volume is recorded in cm^3. The boxes were recorded in the Table A5 in a systematic way. This is of course not the only possible way. Another is shown in Table A6.

Table A5

Systematic Arrangement			
Box	Size	Volume cm^3	Ratio of volume of neighbouring pairs V_1/V_2
A	3,3,3	27	-
B	3,3,5	45	0.6
C	3,3,8	72	0.625
D	3,5,5	75	0.96
E	3,5,8	120	0.625
F	3,8,8	192	0.625
G	5,5,5	125	1.536
H	5,5,8	200	0.625
I	5,8,8	320	0.625
J	8,8,8	512	0.625

Table A6

Arrangement in Order of Volume			
Box	Size	Volume cm^3	Ratio of volume of neighbouring pairs V_1/V_2
A	3,3,3	27	-
B	3,3,5	45	0.6
C	3,3,8	72	0.625
D	3,5,5	75	0.96
E	3,5,8	120	0.625
G	5,5,5	125	0.96
F	3,8,8	192	0.65
H	5,5,8	200	0.96
I	5,8,8	320	0.625
J	8,8,8	512	0.625

Page 12 in the unit

The volume problem can be used as a revision of or as an introduction to ratio. The volumes of the two boxes in the drawing on page 12 are 75 and 72 cm^3. The ratio between these volumes is 75/72 = 1.04. It is difficult to distinguish by eye between these two boxes. The difference between the two volumes is not a significant quantity. For example, consider a box of volume 8 cm^3 and one of 10 cm^3. The difference in their volumes is 2 cm^3, but the ratio of the larger volume to the smaller is 1.25. Consider two boxes of volumes 80 cm^3 and 82 cm^3. Again the difference is 2 cm^3; but the ratio of volumes is 1.025. This second pair of boxes would be difficult to order by eye.

Other problems can be formulated using different triplets, providing a source for calculations of volume and ratio. (For a detailed account of this activity see Kuper and Walter, 1976.)

Page 13 of the unit

Children tend to think that squares are not rectangles. Discuss the idea that the square is a special form of quadrilateral and rectangle.

Three boxes are made from squares only, and one using no squares.

The patterns illustrated on page 13 of the unit are an introduction to the next section and will be examined in detail later.

Pages 14 to 17 of the unit

Provide each child with five identical squares of card and encourage them to move the squares about to find the patterns. These should be drawn on squared paper and cut out. Squares of side length 3 or 4 cm form a suitable size.

The rules for forming the patterns are shown on page 14 of the unit. The students can test for congruent shapes by fitting them on top of each other. Among the patterns which are often duplicated is pattern (a) (Fig. A4).

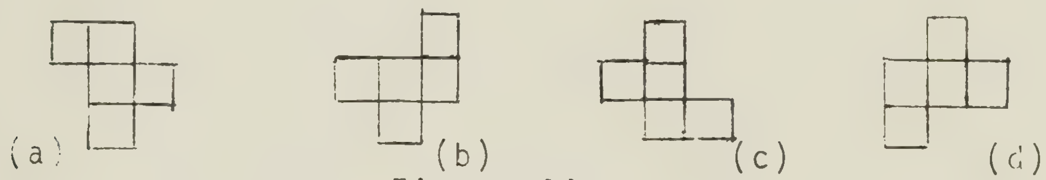


Figure A4

Rotations of some Pentominoes

It can be placed in eight different orientations as shown in Figure 4 (a) - (h). (b), (c) and (d) are found by successive 1/4 turn rotations of (a). Patterns (e), (f), (g) and (h) are obtained by successively "flipping" (a), (b), (c) and (d). Alternatively we can "flip" (a) and then rotate successive 1/4 turns.

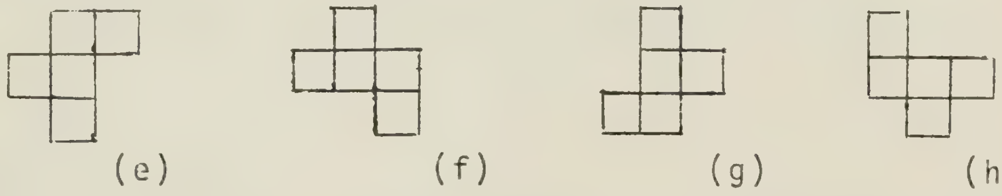
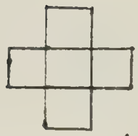


Figure A4

Rotations of some Pentominoes

The pattern (k) is not usually repeated. It is symmetric with respect to 1/4 turns.



(k)

Figure A4

Rotations of some Pentominoes

The duplicate patterns on page 17 of the unit are: A and C, D and G, E and H.

The missing patterns on page 18 are:

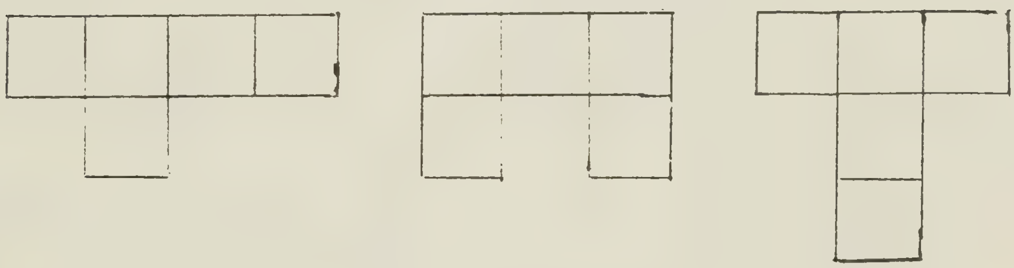


Figure A5

Pentominoes Missing from Page 18 of the Unit

<u>Page 19 of the unit</u>

There are eight patterns that fold into an open box. These are marked B in Figure A6.

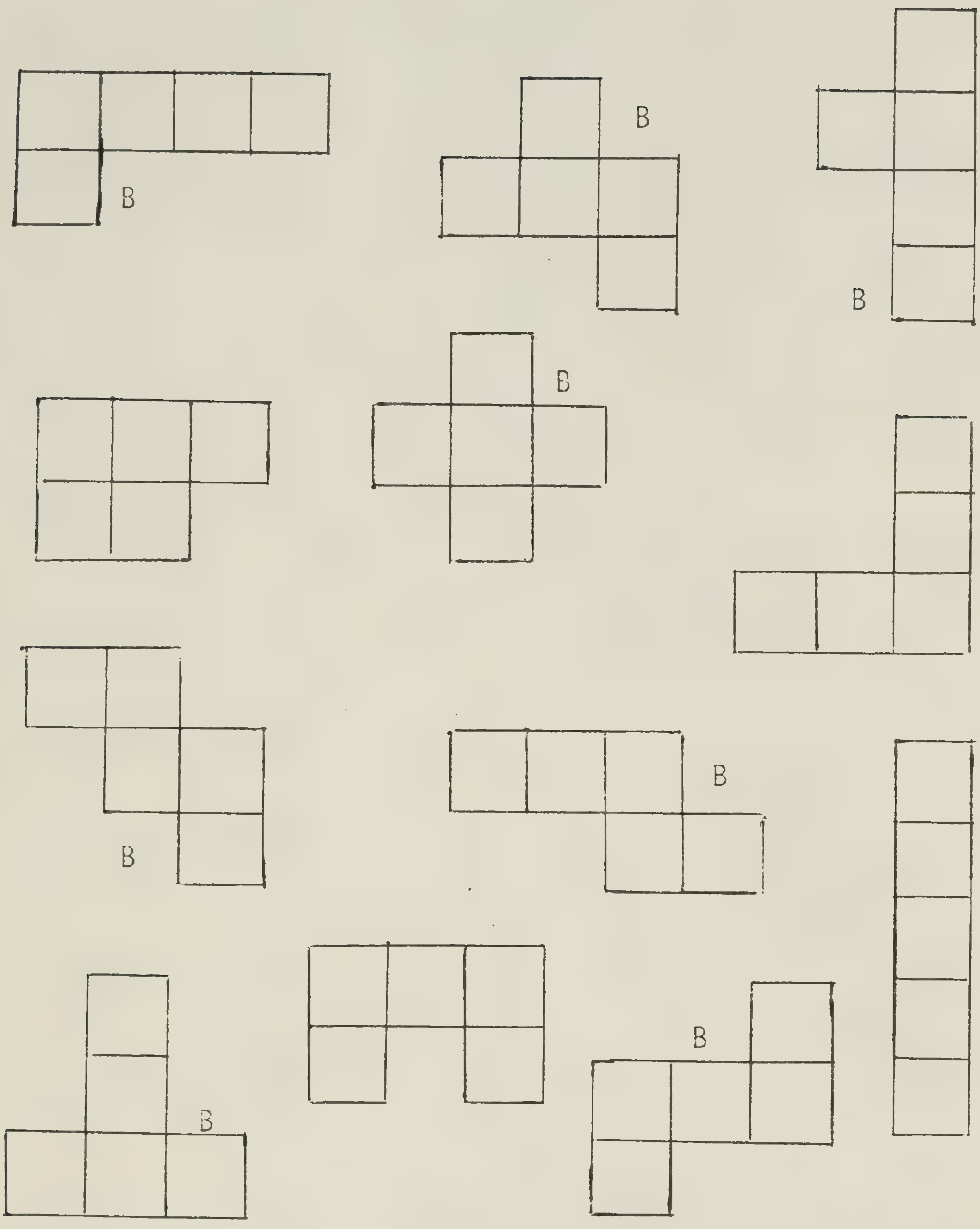


Figure A6

The 12 Pentominoes

Page 20 of the unit

This is an optional exercise, as it is difficult. There are 35 possible patterns of which 12 can be folded into a closed box. Figure A8 shows the 35 possibilities. These form two groups: (a) 11 odd hexominoes (covering 3 black and 3 white squares of a chess board) and (b) 24 even hexominoes (covering four squares of one colour and 2 of the other on the chess board). The patterns which will fold into a closed box are marked B. (Gardener, 1959; Golomb, 1967)

It may be helpful to first discuss the patterns made from 1, 2, 3 and 4 squares, before attempting this problem.

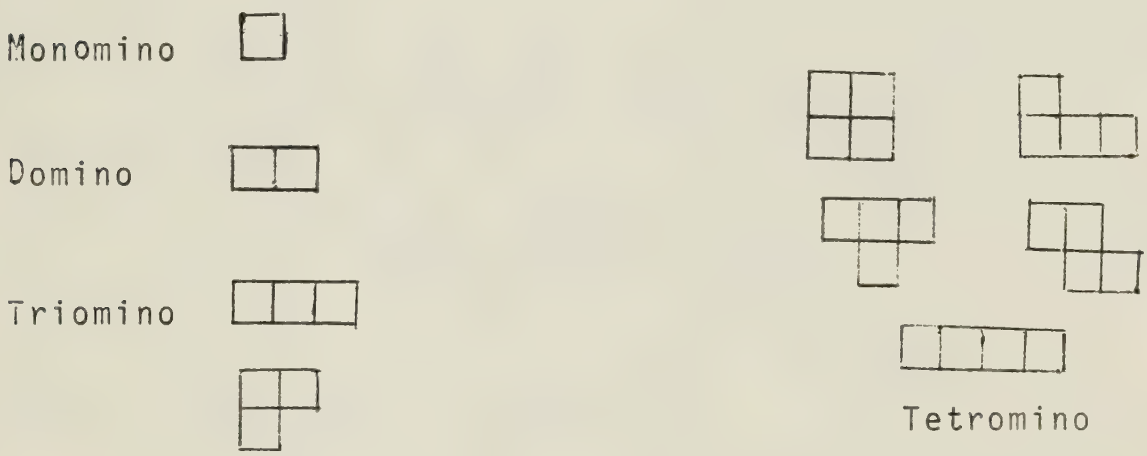
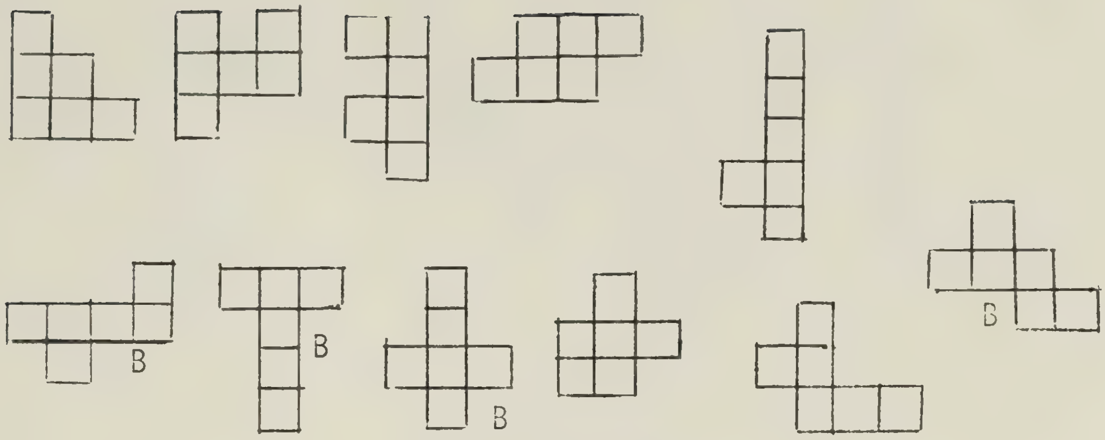


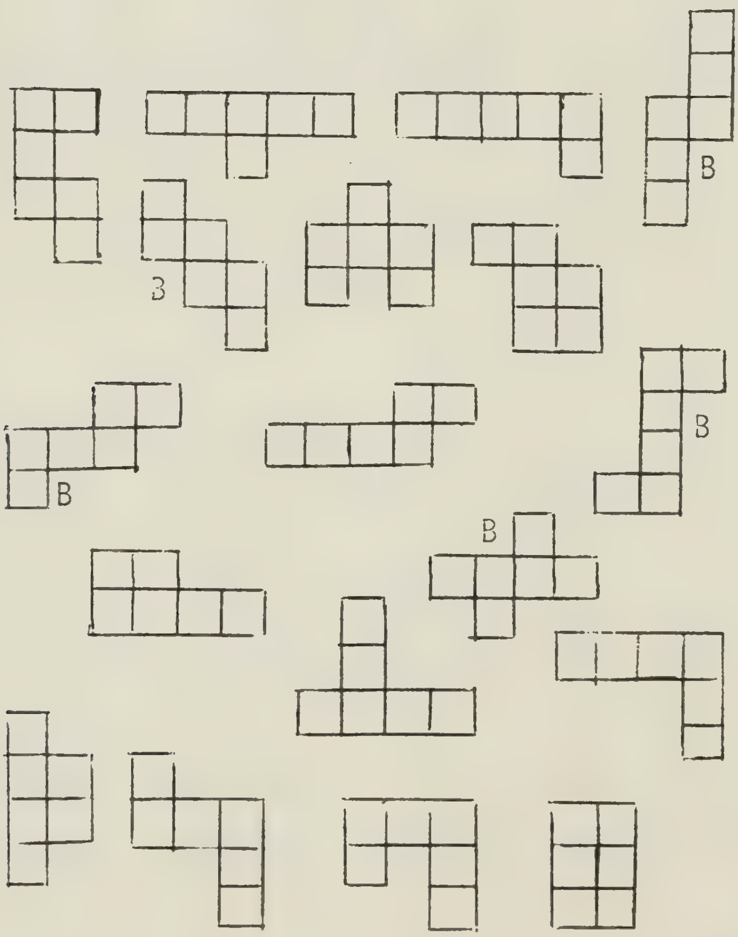
Figure A7

Polyominoes of Order 1, 2, 3, and 4



(The hexominoes which will fold into closed boxes are marked B)

11 "Even" Hexominoes



24 "Odd" Hexominoes

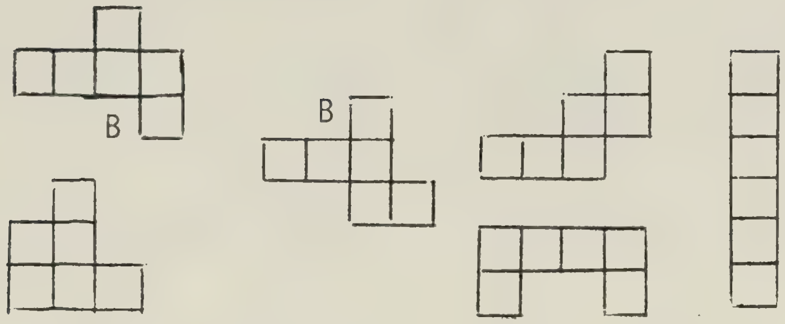


Figure A8

The Hexominoes

Pages 21 and 22 of the unit

The children now make an accurate model of an open box. The net can be copied or drawn with ruler and compasses. Score the fold lines with a sharp instrument to make folding easier.

The flaps can be placed in other places, not necessarily those shown in the diagram.

Pages 23 and 24 in the unit

These pages give the child opportunity to examine the box carefully, and to compare the box with the drawings of the box in the unit. Make sure that the letters are placed in the direction shown on page 23. The right-hand drawing on page 23 is not possible. On page 24 both the right-hand drawing and the bottom one are impossible.

Page 25 in the unit

There are 30 different ways of colouring an open cube,

using five different colours. In order to discover this the child would need a set of more than 30 cubes. However it is sufficient to provide one finished set for discussion with the class at the end of the assignment. This set is made easily by using coloured sticky paper on the sides of the boxes.

However if the child is provided with a duplicated sheet, containing between 40 and 50 of the nets shown in the next diagram, he will be able to investigate the problem by colouring the nets.

To calculate the number of different ways to colour the outside of the open cube using 5 different colours, it is convenient to start by colouring the base. (Figure A9) There are 5 ways of choosing a colour for the base. Now there are four colours remaining. Consider one of the faces, say A. Choose one of the remaining colours for A. Since the open box has fourfold symmetry, we need not count how many ways we can colour face A. The face opposite A can be coloured in 3 different ways. Thus there are $3 \times 5 = 15$ ways of colouring the base and two faces. There are two different ways of colouring the remaining two faces with two colours. Thus there are $5 \times 3 \times 2 = 30$ different ways of colouring the open cube using five different colours.

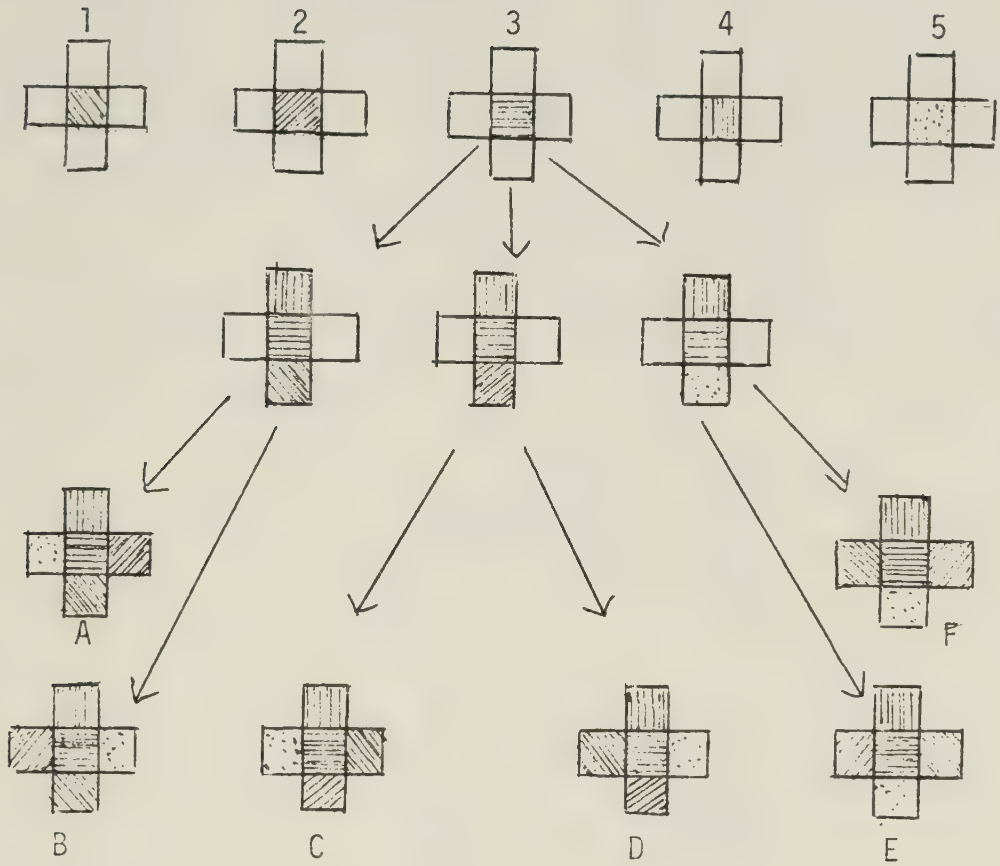


Figure A9

Colouring an Open Cube

The six patterns *A, B, C, D, E,* and *F,* are made from the pattern 3. From each of the other patterns *1, 2, 4,* and *5* we also get *6* different variations of the colouring problem. Thus there are *5* x *6* ways of colouring the open cube.

<u>Pages 26 to 29 in the unit</u>

This exercise is designed to provide experience in looking at lines in a three-dimensional setting.

The longest stick that will fit in is the side length $x\sqrt{3}$. In the case of the cube built in this unit the stick is about 1.7 x 5 cm or about 8.5 cm long. It fits into the

box from A to G, (Figure A10) from B to H, from C to E and from D to F. The sticks are given to the students in whole centimetre lengths. This avoids the discussion about which is the longest. It is clear to the students that the 8 cm rod goes in, but the 9 cm one is too big.

In order to calculate the main diagonal, we use Pythagoras's theorem.

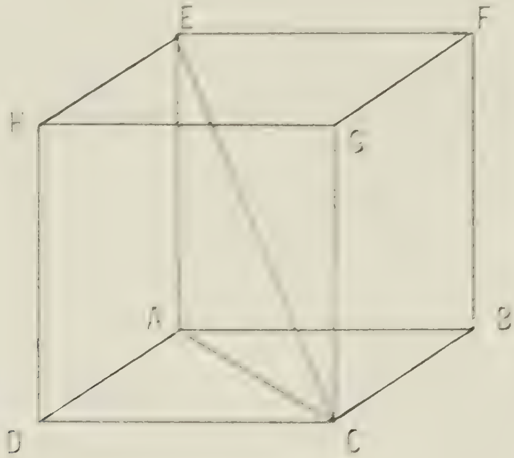


Figure A10

Diagonals of the Cube

We need to calculate the length EC, the main or longest diagonal of the cube. (EC is the same length as AG, DF, BH)

Let the edge length of the box be a cm. In $\Delta\ ADC$, by the Pythagorean theorem, $< ADC = 90^0$

$$AD^2 + DC^2 = AC^2$$

$$a^2 + a^2 = AC^2$$

$$2a^2 = AC^2$$

$$AC = a\sqrt{2}$$

Similarly, in $\Delta\ EAC$, $<EAC = 90^{0}$, therefore

$$AE^2 + AC^2 = EC^2$$

$$a^2 + 2a^2 = EC^2$$

$$3a^2 = EC^2$$

$$EC = a\sqrt{3}$$

The longest stick which will fit on the bottom is AC or DB, i.e. of length $a\sqrt{2}$.

Page 30 in the unit

The ratio of the diagonal to the side length is $1:\sqrt{3}$. This is an optional exercise.

Page 35 of the unit

The rectangle which will just fit into the open cube is the one labelled *B*. It is about 7 cm by 5 cm ($5\sqrt{2}$ by 5 cm).

The rectangle fits into the box in two *different* ways, as shown in the diagram. Altogether the rectangle fits in 6 ways. The 5 cm edge lies on *EA* and *GC*, *HD* and *FB*, *HE* and *CB*, *HG* and *AB*, *GF* and *AD*, or *EF* and *DC*.

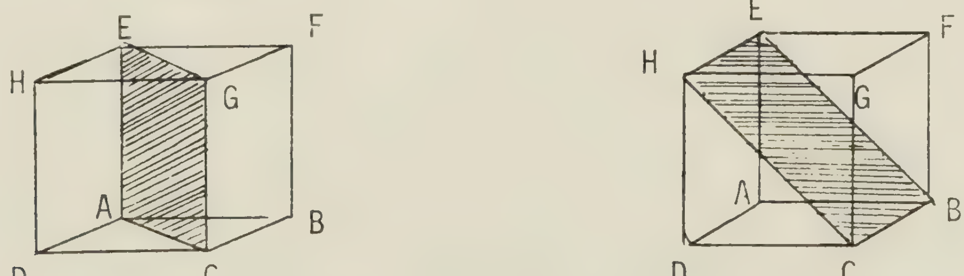


Figure A11

The Rectangle and the Cube

Page 36 in the unit

The stick will lie along the diagonal of the rectangle, and is the same length as the diagonal of the cube, namely $5\sqrt{3}$ cm.

Pages 37 and 38 of the unit

These two pages are optional.

Additional problems

These problems have not been arranged in any particular order. They include ideas which were not used in the unit, in order not to increase the length of the unit itself. More ideas can be found in Kuper and Walter (1976).

(1) The work can be carried out with other lengths.

(2) Suppose the volume of a particular box were halved. What should one do to the original volume to get the new volume? For example, the 3, 5, 8 box has a volume of 120 cm^3. If one made the box 5, 3, 4 one has a box of volume 60 cm^3. Try similar problems.

(3) The Pythagorean problem can be done in greater detail.

(4) Discuss the problem of finding boxes of the same volume, using different rectangles.

(5) Investigate the "stick" problem with the boxes used in (2) above.

(6) Calculate the surface areas of the boxes, and the ratio of the surface area of neighbouring boxes as described on page 11 of the unit.

(7) Calculate the ratio surface area/volume or volume/surface area. Are there interesting results? When is the ratio a maximum or minimum?

(8) Make an open and a closed box, the open box being slightly bigger than the closed box, so that the latter will fit easily into the former. Using the

closed box, repeat the colouring problem of page 25 of the unit, using six different colours.

(9) How many differnt ways can one put the coloured closed cube into the coloured open cube?

(10) How many different ways can one colour a box using 2, 3, 4, 5, or 6 different colours?

Materials required for the unit and teaching time

Stage 1. Pages 1 to 7	Approximate number of lessons
15 pieces each of card: 3 by 3, 3 by 5, 3 by 8, 5 by 5, 5 by 8, 8 by 8.	2
Masking tape	
Scissors	
Plastic bag (25 cm by 35 cm) for storage	
Notebook	
Stage 2. Pages 8 to 12	2
The set of boxes made by the children in the Stage 1	
Sets of strongly made boxes	
Cubes whose edge is one centimetre, or Cuisenaire rods	
Stage 3. Page 13 to 19	2
For each child - 5 squares of card, side 3 cm; 1 sheet of construction paper or squared paper; Scissors and glue	
A demonstration set of the 12 pentominoes, scored along the fold lines	
Possible poster displays - Tessellations of the pentominoes; Tessellations of the hexominoes; Diagram of all the 12 pentominoes fitted together to form a rectangle	

	Approximate number of lessons
<u>Stage 4. Pages 20 to 25</u>	2 to 3
Squared paper	
Net of the box on page 21	
Construction paper	
Scissors, coloured pens or pencils, glue	
Duplicated sheet containing at least 40 small copies of the net on page 21	
<u>Stage 5. Pages 26 to 30</u>	2
The open box made in stage 4	
Sticks or pieces of drinking straws of length 45 to 50 cm	
Scissors	
Nets of boxes of other edge lengths	
<u>Stage 6. Pages 35 to 38</u>	1
Rectangles of card: 3 by 5, 5 by 10 5 by 7, 5 by 8½	
The sticks used in stage 5, and the box made in stage 4	
Plasticine, glue and scissors	

<u>To the teacher</u>: (Supplied to teachers in Edmonton)

This folder contains the following material:

1. Booklet "Making Rectangular Solids" by Marie Kuper and Marion Walter. (Please note that this material is copyrighted and may not be reproduced without permission.)
2. List of materials required for each section.
3. Reproduction of a series of lessons given by Marie Kuper at the Talmud Torah School, Edmonton, in Grade 6. The material was taught in 9 lessons of 55 minutes. It is preferable to spend 12 lessons of 45 to 50 minutes. So some points were omitted and curtailed in the protocol, which is intended as a guide only. Comments will be most welcome.
4. Teachers' guide (written before the unit had been taught).
5. Article "Edges to Solids" written by Marie Kuper and Marion Walter, explaining the mathematics in the unit.
6. Proposal for this study, presented to Ed CI 596, at the University of Edmonton, March 1975.